Towards non-hydrostatic spectral models

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Motivations

- Spectral models are used at leading NWP centres.

- Current semi-Lagrangian cores do not make the most out of spectral methods.

- Non-hydrostatic effects need to be evaluated with the non-hydrostatic version rather than a different model.
Three keys

• Accurate associated Legendre functions at high order and degree

• Accurate interpolation in semi-Lagrangian advection

• Stable and simple non-hydrostatic formulation
Assoicated Legendre functions

Enomoto et al. (2004; 2008);
Enomoto and Miyamoto, MSJ 2012 Spring
Conventional

Swarztrauber (1993)
Conventional three-term recurrence

\[
\hat{P}_n^m = (a_n^m \cos \theta) \hat{P}_{n-1}^m - b_n^m \hat{P}_{n-2}^m \quad (1)
\]

\[
\hat{P}_m^m = (d_n^m \sin \theta) \hat{P}_{m-1}^{m-1} \quad (2)
\]

\[
\hat{P}_{m+1}^m = (a_{m+1}^m \cos \theta) \hat{P}_m^m \quad (3)
\]

where \( \theta \) is the colatitude and

\[
a_n^m = \sqrt{\frac{4n^2 - 1}{n^2 - m^2}}, \quad b_n^m = \frac{a_n^m}{a_{n-1}^m}, \quad d_n^m = \sqrt{\frac{2m + 1}{2m}} \quad (4)
\]
The alternative four-term recurrence

involves \((m-2,n), (m-2,n-2), \text{ and } (m-2, n-2)\) to calculate the value at \((m,n)\):

\[
\tilde{P}_n^m = \sqrt{\frac{(2n + 1)(n + m - 2)(n + m - 3)}{(2n - 3)(n + m - 1)(n + m)}} \tilde{P}_n^{m-2} \\
- \sqrt{\frac{(n - m + 1)(n - m + 2)}{(n + m - 1)(n + m)}} \tilde{P}_n^{m-2} \\
+ \sqrt{\frac{(2n + 1)(n - m)(n - m - 1)}{(2n - 3)(n + m - 1)(n + m)}} \tilde{P}_n^{m-2}
\]  

(5)
Why does the three-term recurrence fail?

- A floating-point number has can only represent a certain small number e.g. $x_{\text{min}} = 2.23 \times 10^{-308}$ for double precision.

- $\tilde{P}_m^m \propto \sin^m \theta$ is very small near the poles at high orders.

- $|\tilde{P}_n^m|$ grows with degree $n$. 
Legendre synthesis–analysis test at T2159

Conventional

Swarztrauber (1993)
Legendre synthesis–analysis test

maximum error

truncation wavenumber
Extended-range arithmetic

- Smith et al. (1981) avoided overflow and underflow in the evaluation of associated Legendre functions with extended-range arithmetic at the 2x computational cost.

- Fukushima (2011) proposed an accelerated version that enables to evaluate extremely high degree as $2^{32}$ at only 10% increase in computational time.

Express $X$ with a pair of a floating-point number $x$ and an integer $i$.

$$X = xB^i$$  \hspace{1cm} (6)

where $B$ is a large power of 2.
**Validation of associated Legendre functions calculated from the four-term recurrence**

- The check sum
  \[ \int_{-1}^{1} [\tilde{P}_n^m]^2 \, dx = 1 \]  
  (7)
  shows that the Swarztrauber (1993)'s recurrence enables accurate Legendre transforms at high order and degree.

- Associated Legendre functions evaluated by the method of Fukushima (2011) is regarded as the truth.

- The small values from the four-term recurrence is found to be in inaccurate.
Summary for associated Legendre functions

- The four-term recurrence (Swarztrauber 1993) is stable over 2000.

- The conventional three-term recurrence may be used in extended-range arithmetic to avoid overflow and underflow (Smith et al. 1981).

- An optimized evaluation with extended-range arithmetic only require 10% increase in computational time (Fukushima 2011).

- Associated Legendre functions from the four-term recurrence are validated with those from the three-term recurrence using extended-range arithmetic to be inaccurate at small values.
Interpolation
Enomoto 2008;
Lauritzen et al. 2012, in preparation
Advection

- Eulerian advection
  - is dispersive
  - requires a short time step.

- Semi-Lagrangian advection
  - allows a longer time step
  - needs interpolation, which determines the accuracy
  - is typically diffusive

→Semi-Lagrangian advection with an accurate interpolation should be less dispersive and diffusive.
Quasi-cubic interpolation

Ritchie et al. (1995)
Bicubic interpolation

Derivatives calculated in spectral space (Enomoto 2008)
Rotation of a Gaussian hill

Enomoto (2008)
Standard test suite

(a) \( \phi \,(t=0) \), Gaussian hills
(b) \( \phi \,(t=0) \), cosine bells
(c) \( \phi \,(t=0) \), slotted cylinders
(d) \( \phi \,(t=0) \), 'correlated' cosine bells

Lauritzen and Skamarock (2010)
Convergence

nofilter

$l_2$ error norm, cosine hills

filter+fix

$l_2$ error norm, cosine hills

3 1.5 0.75 0.375 0.1875°

T119 CFL 5.2 $l_2$ error norm cos hills
## ‘Minimal’ resolution compared

<table>
<thead>
<tr>
<th>model</th>
<th>reference</th>
<th>filter</th>
<th>Ne</th>
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</thead>
<tbody>
<tr>
<td>Quasi-cubic</td>
<td>Ritchie et al. 1995</td>
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<td>Spectral bicubic</td>
<td>Enomoto 2008</td>
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</tr>
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<td></td>
<td></td>
<td>yes</td>
<td>2.29</td>
</tr>
</tbody>
</table>

obtained from convergence of $\ell_2$ error norm cos hills
Summary for interpolation

Spectral bicubic interpolation

• is easy to implement in existing spectral models.

• is accurate for smooth and non-smooth tracers.

• generates ripples but they can be removed with a short-wave filter of Sun et al. (1996).
Non-hydrostatic formulation
Enomoto and Juang, MSJ 2011 Autumn
\(\sigma\)-co-ordinates in hydrostatic pressure \(\overline{p}\)  
(Laprise 1992)

- Additional prognostic variables: \(w\) and \(p\)
- Monotone in the vertical unlike full pressure \(p\) (Miller 1974)
- No additional complex metric terms unlike terrain following height co-ordinates
- Easy to conserve total energy
- Adopted by Météo France ALADIN (Bubnová 1995)/Arpège (Yessard 2008), ECMWF NH-IFS (Wedi et al. 2009) and JMA GSM (Yoshimura, JMSJ spring 2011 B401)
Non-hydrostatic double-fourier version of JMA GSM (Yoshimura)

TL1279L60, $\Delta t=10$ min, FT = 2 d, slp (hPa)

Spherical harmonics

Double Fourier Series
Non-hydrostatic double-fourier version of JMA GSM (Yoshimura)

TL1279L60, $\Delta t=10$ min, $FT = 2$ d, precipitation (mm/d)

Spherical harmonics

Double Fourier Series
Non-hydrostatic IFS at ECMWF (Wedi)

20101015 12Z + 48h
You already have a gem at NCEP.
NCEP MSM (Juang 1992; 2000)

- is a non-hydrostatic version of NCEP RSM (Juang and Kanamitsu 1994).

- uses horizontal discretization by double Fourier series.

- uses the perturbation method.

- shares the same physics packages with NCEP GSM.

- uses the Euler equations transformed from $z$- to $\sigma$- co-ordinates.
Pressure gradient terms

Laprise (1992)

$$\frac{1}{\rho} \nabla z p = RT \nabla_\sigma \ln p + \frac{p \partial \ln p}{\bar{p} \partial \ln \sigma} \nabla_\sigma \phi, \frac{\partial \phi}{\partial \sigma} = -\frac{RT \bar{p}_s}{\rho} \quad (8)$$

Juang (1992)

$$\frac{1}{\rho} \nabla z p = RT \nabla_\sigma \ln p + \frac{T \partial \ln p}{\bar{T} \partial \ln \sigma} \nabla_\sigma \phi, \frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma} \quad (9)$$
The hydrostatic state of Laprise (1992)

Define

\[
\frac{\partial \phi}{\partial \bar{p}} = -\frac{1}{\bar{\rho}},
\]

(10)

means

\[
\bar{\rho} = \rho.
\]

(11)

Since \(\bar{\rho} = \bar{p}/R\bar{T}, \rho = p/RT,\)

\[
\frac{T}{\bar{T}} = \frac{p}{\bar{p}}
\]

(12)
Hydrostatic variables

Choice of the hydrostatic temperature $\bar{T}$ and surface pressure $\bar{p}_s$:

1. Time independent (Juang 1994; Gallus and Rančić 1996)

2. Both $\bar{T}$ and $\bar{p}_s$ are determined externally (Juang 1992).

3. Impose $\bar{T}$ externally but predict $\bar{p}_s$ internally (Juang 2000, default of MSM).

4. Predict both $\bar{T}$ and $\bar{p}_s$ internally (Juang 2000).
The unified system by Arakawa and Konor (2009)

- unifies anelastic and primitive (quasi-hydrostatic) systems.
- uses the hydrostatic density $\bar{\rho}$ in the continuity equation.
  - by ignoring non-hydrostatic pressure tendency $\partial (p - \bar{p})/\partial t$: $p - \bar{p}$ is obtained by solving an Helmholtz equation.
- removes sound waves without distortion of planetary waves.
Non-hydrostatic spectral models

Sound and gravity waves

Fully-Compressible

Unified

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Non-hydrostatic spectral models

Enomoto, Miyamoto and Juang

Sound and gravity waves

Fully-Compressible

Quasi-Hydrostatic

$|\omega| s^{-1}$

$L_x \text{ km}$

$|\omega| s^{-1}$

$L_x \text{ km}$

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Non-hydrostatic spectral models

Rossby waves

Enomoto, Miyamoto and Juang

Fully-Compressible, Unified, Quasi-Hydrostatic

Pseudo-Incompressible

Lx km

\(|\omega| \text{ s}^{-1}\)

\(|\Omega| \text{ s}^{-1}\)

\(|\Omega| \text{ s}^{-1}\)
The hydrostatic state of Arakawa and Konor (2009)

Using the hydrostatic Exner function $\bar{\pi} = (\bar{p}/p_{\text{ref}})^\kappa$, define

$$
\frac{\partial \bar{\pi}}{\partial z} \equiv -\frac{g}{c_p \bar{\theta}}, \quad (13)
$$

which means

$$
\bar{\theta} = \theta. \quad (14)
$$

Since $\theta = T/\pi$ and $\bar{\theta} = \bar{T}/\bar{\pi}$, hydrostatic temperature $\bar{T}$ may be written as

$$
\bar{T} = T \left(\frac{\bar{p}}{p}\right)^\kappa. \quad (15)
$$
Forecast experiment using NCEP MSM

- Initial time: 0 UTC 11 August 2011, 24-hour forecast
- Horizontal resolution: $\Delta h = 26$ km, vertical levels: 42, time step: $\Delta t = 60$ s
- Initial and boundary conditions: NCEP GFS
- Projection: polar stereo
- Domain: (132–141E, 31–39N) centred at (136E, 35N)
- Problem size: $32 \times 32 \times 42$
Non-hydrostatic spectral models

slp hPa $FT=24$ $INIT=2011081800$

Initial state

Unified

Boundary (default)

Laprise
Summary for non-hydrostatic formulation

• The governing equation of MSM is the Euler equations in $\sigma$–co-ordinates transformed from those in $z$–co-ordinates (Juang 1992).

• Sound wave can be removed without distorting planetary wave by the use of the hydrostatic density $\bar{\rho}$ in the continuity equation (Arakawa and Konor 2009).

• The hydrostatic temperature state $\bar{T}$ may be defined by the hydrostatic assumption of Arakawa and Konor (2009) or Laprise (1992).

• Preliminary forecast experiments shows that both hydrostatic assumption are stable.
The perturbation method

predicts deviations from from the global model.

\[
\frac{\partial A'}{\partial t} = \frac{\partial A}{\partial t} - \frac{\partial A_b}{\partial t}
\]
The governing equations

\[ \frac{\partial v'}{\partial t} = -m^2 v \cdot \nabla v - \dot{\sigma} \frac{\partial v}{\partial \sigma} - E \nabla m^2 + f k \times v 
- R(\overline{T} + T') \nabla (\overline{Q}_s + Q') \]

\[ - \left(1 + \frac{T'}{T}\right) \left(1 + \frac{\partial Q'}{\partial \ln \sigma}\right) \nabla \overline{\phi} + F - \frac{\partial v_b}{\partial t} \]  \hspace{1cm} (17)

\[ \frac{\partial w'}{\partial t} = -m^2 u \cdot \nabla w - \dot{\sigma} \frac{\partial w}{\partial \sigma} 
- g \left[1 - \left(1 + \frac{T'}{T}\right) \left(1 + \frac{\partial Q'}{\partial \ln \sigma}\right)\right] + F_z - \frac{\partial w_b}{\partial t} \]  \hspace{1cm} (18)
\[
\frac{\partial Q_s'}{\partial t} = -m^2 \int_0^1 \left[ u \frac{\partial Q_s}{\partial x} + v \frac{\partial Q_s}{\partial y} + \left( \left. \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] d\sigma - \frac{\partial Q_{sb}'}{\partial t} \tag{19}
\]

\[
\frac{\partial Q'}{\partial t} = -m^2 u \frac{\partial Q_s'}{\partial x} - m^2 v \frac{\partial Q_s'}{\partial y} - \dot{\sigma} \frac{\partial Q'}{\partial \sigma} - \frac{\dot{\sigma}}{\sigma} \frac{\partial Q_s'}{\partial \sigma} - \gamma \nabla_3 \cdot \mathbf{v} + \gamma \frac{F_T}{T} - \frac{\partial Q_s'}{\partial t} \tag{20}
\]

\[
\frac{\partial T'}{\partial t} = -m^2 u \frac{\partial T'}{\partial x} - m^2 v \frac{\partial T'}{\partial y} - \dot{\sigma} \dot{\sigma} \frac{\partial (T + T')}{\partial \sigma} - \kappa \frac{\partial (T + T')}{\partial \sigma} - \frac{RT}{c_v} \nabla_3 \cdot \mathbf{v} + F_T - \frac{\partial T_{b}'}{\partial t} \tag{21}
\]

\[
\frac{\partial q'}{\partial t} = -m^2 u \frac{\partial q}{\partial x} + m^2 v \frac{\partial q}{\partial y} - \dot{\sigma} q \sigma + F_q - \frac{\partial q_{b}'}{\partial t} \tag{22}
\]
Vertical velocity and divergence

\[ \dot{\sigma} = \frac{\sigma}{RT} \left[ gw + \frac{\partial \bar{\phi}}{\partial t} + m^2 \left( u \frac{\partial \bar{\phi}}{\partial x} + v \frac{\partial \bar{\phi}}{\partial y} \right) \right] \]  

(23)

\[ \nabla_3 \cdot \mathbf{v} = m^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\sigma}{RT} \left[ m^2 \left( \frac{\partial u \partial \bar{\phi}}{\partial \sigma \partial x} + \frac{\partial v \partial \bar{\phi}}{\partial \sigma \partial y} \right) - g \frac{\partial w}{\partial \sigma} \right] \]  

(24)

\[ \bar{\psi} = \bar{\psi}_s + \int_{\sigma}^{\sigma_s=1} \frac{RT}{\sigma} d\sigma \]  

(25)
\[ Q' \equiv \ln p - \ln \bar{p} = Q - \bar{Q}_s - \ln \sigma \] (26)

gives
\[ \left( \frac{\bar{p}}{p} \right)^\kappa = \exp(-\kappa Q'). \] (27)

Using this identity
\[
\nabla T = \left( \frac{\bar{p}}{p} \right)^\kappa (\nabla T - \kappa T \nabla Q')
\]
\[ = \exp(-\kappa Q') (\nabla T - \kappa T \nabla Q') . \] (28)

\[ \nabla T \] is used in \( \dot{\sigma} \) and \( \nabla_3 \cdot \mathbf{v} \).