Semi - Lagrange Notes

These notes are intended for use by coders. The equations are written using non - standard multi - character symbols which appear in the computer code. These equations have first been derived using vector notation and subsequently expanded for detailed coding.

Stable extrapolation two time level scheme
SETTLS

$$\vec{R}_A^{t+\Delta t} \approx \vec{R}_D^{t} + \Delta t \cdot \left[\frac{d}{dt}\vec{R}\right]_D^{t} + \frac{\Delta t^2}{2} \cdot \left[\frac{d^2}{dt^2}\vec{R}\right]_M^{t+\frac{\Delta t}{2}}$$

↓

$$\vec{R}_A^{t+\Delta t} \approx \vec{R}_D^{t} + \Delta t \cdot \vec{V}_D^{t} + \frac{\Delta t^2}{2} \cdot \left[\frac{d}{dt}\vec{V}\right]_M^{t+\frac{\Delta t}{2}}$$

$$\left[\frac{d}{dt}\vec{V}\right]_M^{t+\frac{\Delta t}{2}} = \ ?$$

$$\left[\frac{d}{dt}\vec{V}\right]_M^{t+\frac{\Delta t}{2}} \approx \left[\frac{d}{dt}\vec{V}\right]_M^{t-\frac{\Delta t}{2}} = \frac{\vec{V}_A^{t} - \vec{V}_D^{t-\Delta t}}{\Delta t}$$

$$\vec{R}_A^{t+\Delta t} = \vec{R}_D^{t} + \frac{\Delta t}{2} \cdot \left(\left[2\vec{V}^{t} - \vec{V}^{t-\Delta t}\right]_D + \vec{V}_A^{t}\right)$$

Two Time Level Semi-Lagrangian Semi-implicit With SETTLS

$$\frac{dX}{dt} = R = N + L$$

$$\text{explicit} \quad \frac{\left(X^A_{t+\Delta t} - X^D_t\right)}{\Delta t} = N^M_{t+\frac{\Delta t}{2}} + L^M_{t+\frac{\Delta t}{2}}$$

$$\text{semi-implicit} \quad \frac{\left(X^A_{t+\Delta t} - X^D_t\right)}{\Delta t} = N^M_{t+\frac{\Delta t}{2}} + \frac{1}{2}\left(L^A_{t+\Delta t} + L^D_t\right)$$

$$= N^M_{t+\frac{\Delta t}{2}} + \left(L^M_{t+\frac{\Delta t}{2}} - L^M_{t+\frac{\Delta t}{2}}\right) + \frac{1}{2}\left(L^A_{t+\Delta t} + L^D_t\right)$$

$$= N^M_{t+\frac{\Delta t}{2}} + L^M_{t+\frac{\Delta t}{2}} + \frac{1}{2}\left[L^A_{t+\Delta t} - 2L^M_{t+\frac{\Delta t}{2}} + L^D_t\right]$$

$$= \text{explicit} + \text{correction}$$

$$\text{correction} = \frac{1}{2}\left[L^A_{t+\Delta t} - 2L^M_{t+\frac{\Delta t}{2}} + L^D_t\right] = \frac{1}{2}\Delta_{tt}L$$

The correction can be viewed as semi-Lagrange Laplacian.

The correction can also be weighted with a factor β.

$$\frac{dX}{dt} = (N + L)^M_{t+\frac{\Delta t}{2}} + \frac{1}{2}\beta\Delta_{tt}L$$

The terms of the form $F^{M}_{t+\frac{\Delta t}{2}}$ are approximated :

$$F^{M}_{t+\frac{\Delta t}{2}} \approx \frac{1}{2}\left(F_t^A + \left(2F_t - F_{t-\Delta t}\right)^D\right)$$

(ECMWF Settls approximation)

Let $\dot{X} = N + L$

$$X^{A}_{t+\Delta t} = X_t^D + \frac{\Delta t}{2}\left(\dot{X}_t^A + \left(2\dot{X}_t - \dot{X}_{t-\Delta t}\right)^D\right) + \frac{1}{2}\beta\Delta t\Delta_{tt}L$$

$$X^{A}_{t+\Delta t} = \frac{\Delta t}{2}\dot{X}_t^A + \Delta t\left[\frac{X_t}{\Delta t} + \dot{X}_t - \frac{1}{2}\dot{X}_{t-\Delta t}\right]^D + \frac{1}{2}\beta\Delta t\Delta_{tt}L$$

expand :

$$\Delta_{tt}L = \left[L^{A}_{t+\Delta t} - 2L^{M}_{t+\frac{\Delta t}{2}} + L_t^D\right]$$

$$= L^{A}_{t+\Delta t} - 2\left\{\frac{1}{2}\left[L_t^A + \left(2L_t - L_{t-\Delta t}\right)^D\right]\right\} + L_t^D$$

$$\Delta_{tt}L = L^{A}_{t+\Delta t} - L_t^A + \left(L_{t-\Delta t} - L_t\right)^D$$

The surface pressure equation.

We need an expression for $\frac{\mathrm{d}p_s}{\mathrm{d}t}$.

start with continuity:

$$\frac{\partial}{\partial t}\frac{\partial p}{\partial \eta} + \nabla_3 \cdot \frac{\partial p}{\partial \eta} \vec{V}_3 = 0 \text{, expand and regroup:} \qquad 1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial \eta}\left(D + \frac{\partial \dot{\eta}}{\partial \eta}\right), \qquad D = \nabla \cdot \vec{\mathrm{v}}_H \qquad 2$$

now also express $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial p}{\partial \eta}$ using $p = A(\eta) + B(\eta)p_s$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial p}{\partial \eta} =$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \eta}\left(A + Bp_s\right) + \vec{\mathrm{V}} \cdot \nabla \frac{\partial}{\partial \eta}\left(A + Bp_s\right) + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta}$$

$$= \frac{\partial B}{\partial \eta}\frac{\partial p_s}{\partial t} + \vec{\mathrm{V}} \cdot \nabla \frac{\partial B}{\partial \eta} p_s + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta}$$

$$= \quad \frac{\partial B}{\partial \eta}\left(\frac{\partial p_s}{\partial t} + \vec{\mathrm{V}} \cdot \nabla p_s + \dot{\eta}\,\frac{\partial p_s}{\partial \eta} \right) \quad + \quad \dot{\eta}\,\frac{\partial}{\partial \eta}\,\frac{\partial p}{\partial \eta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial p}{\partial \eta} \quad = \quad \frac{\partial B}{\partial \eta}\,\frac{\mathrm{d}\,p_s}{\mathrm{d}t} \quad + \quad \dot{\eta}\,\frac{\partial}{\partial \eta}\,\frac{\partial p}{\partial \eta}$$

Equate the two expressions for $\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial p}{\partial \eta}$

$$\frac{\partial B}{\partial \eta}\,\frac{\mathrm{d}\,p_s}{\mathrm{d}t} \quad + \quad \dot{\eta}\,\frac{\partial}{\partial \eta}\,\frac{\partial p}{\partial \eta} \quad = \quad -\,\frac{\partial p}{\partial \eta}\left(D + \frac{\partial\,\dot{\eta}}{\partial\,\eta} \right) \qquad 3$$

$$\frac{\partial B}{\partial \eta}\,\frac{\mathrm{d}\,p_s}{\mathrm{d}t} \quad + \quad \frac{\partial p}{\partial \eta}\,D \quad + \quad \frac{\partial}{\partial \eta}\left(\dot{\eta}\,\frac{\partial p}{\partial \eta} \right) \quad = \quad 0$$

Vertical discretization yields :

$$\Delta B_k \frac{\mathrm{d}\, p_s}{\mathrm{d}t} + \Delta p_k D_k + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} - \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}} = 0 \qquad 4$$

for $k = 1, \ldots, K$. Recall that $\sum_{k=1}^{K} \Delta B_k = 1$

Divide eq. 4 by p_s to get $\frac{\mathrm{d}}{\mathrm{d}t} \ln p_s$

$$\Delta B_k \frac{\mathrm{d} \ln p_s}{\mathrm{d}t} + \frac{1}{p_s}\left[\Delta p_k D_k + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} - \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}}\right] = 0 \qquad 5$$

From the vertical finite difference :

$$\left(\dot{\eta}\,\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} =$$

$$-\,p_s \left\{ B_{k+\frac{1}{2}} \frac{\partial q}{\partial t} + \sum_{j=1}^{k} \left[\frac{1}{p_s} D_j \Delta p_j + C_j \Delta B_j \right] \right\} \qquad 6$$

where $\quad q = \ln p_s \qquad C_j = \vec{\mathrm{V}}_j \cdot \nabla \ln p_s$

Substitute (6) in 5 :

$$\Delta B_k \frac{\mathrm{dq}}{\mathrm{d}t} + \frac{1}{p_s} \Delta p_k D_k - \left[B_{k+\frac{1}{2}} \frac{\partial q}{\partial t} + \sum_{j=1}^{k} \quad \right]$$

$$+ \left[B_{k-\frac{1}{2}} \frac{\partial q}{\partial t} + \sum_{j=1}^{k-1} \quad \right] = 0$$

which simplifies to :

$$\Delta B_k \frac{\mathrm{dq}}{\mathrm{d}t} + \frac{1}{p_s} \Delta p_k D_k - \left(B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} \right) \frac{\partial q}{\partial t}$$

$$- \frac{1}{p_s} D_k \Delta p_k - C_k \Delta B_k \quad = \quad 0$$

$$\Delta B_k \frac{\mathrm{d}q}{\mathrm{d}t} \quad - \quad \Delta B_k \left(\frac{\partial q}{\partial t} + C_k \right) \quad = \quad 0 \qquad 7$$

Here $$\frac{\partial q}{\partial t} \quad = \quad - \sum_{k=1}^{K} \left(\frac{1}{p_s} D_k \Delta p_k + C_k \Delta B_k \right) \qquad 8$$

as required by the vertical finite differencing.

We now discretize in time while adding the correction term at every level. The final prediction eq. for the surface pressure is:

$$q_{t+\Delta t}^{A} = \sum_{k=1}^{K} \Delta B_k \left\{ q_t^D + \Delta t \left[Z(k) \right]_{t+\frac{\Delta t}{2}}^{M} - \Delta t \frac{\beta}{2} \Delta_{tt} L \right\} \qquad 9$$

Here $Z(k) = \frac{\partial q}{\partial t} + C_k$, $\frac{\partial q}{\partial t}$ is given by eq. 8

$$L = \frac{1}{p_S^r} \sum_{j=1}^{levs} \left(\Delta p_j^r D_j \right)$$

Evaluation of the RHS of eq. 9:

The $Z(k)$ term:

$$\Delta t \left[Z(k) \right]_{t+\frac{\Delta t}{2}}^{M} = \frac{\Delta t}{2} \left[Z(k)_t^A + \left(2Z(k)_t - Z(k)_{t-\Delta t} \right)^D \right]$$

$$= \frac{\Delta t}{2} Z(k)_t^A + \Delta t \left[Z(k)_t - \frac{1}{2} Z(k)_{t-\Delta t} \right]^D$$

The correction term:

$$-\frac{\beta}{2}\Delta t \Delta_{tt}\left[\frac{1}{p_s^r}\sum_{j=1}^{K}\Delta p_j^r D_j\right] = -\Delta t \Delta_{tt}\delta$$

where $$\delta = \frac{\beta}{2p_s^r}\sum_{j=1}^{K}\Delta p_j^r D_j = S\cdot\tilde{D}$$

$$S = \frac{\beta}{2p_s^r}\left(\Delta p_1^r \quad \dots \quad \Delta p_K^r\right), \quad D = \left(D_1 \quad \dots \quad D_K\right)$$

Note that after the vertical summation δ is a 2 dimensional field.

Since: $$\Delta_{tt}\delta = \delta_{t+\Delta t}^{A} - 2\delta_{t+\frac{\Delta t}{2}}^{M} + \delta_t^D$$

and: $$\delta_{t+\frac{\Delta t}{2}}^{M} = \frac{1}{2}\left[\delta_t^A + \left(2\delta_t - \delta_{t-\Delta t}\right)^D\right]$$

$$\Delta_{tt}\delta = \delta_{t+\Delta t}^{A} - 2\cdot\frac{1}{2}\left[\delta_t^A + \left(2\delta_t - \delta_{t-\Delta t}\right)^D\right] + \delta_t^D$$

$$\Delta_{tt}\delta = \delta_{t+\Delta t}^{A} - \delta_t^A + \left(\delta_{t-\Delta t} - \delta_t\right)^D$$

The final form of eq. 9 is therefore:

$$q_{t+\Delta t}^{A} = \sum_{k=1}^{K} \Delta B_k \left\{ q_t^D + \frac{\Delta t}{2} Z(k)_t^A + \Delta t \left[Z(k)_t - \frac{1}{2} Z(k)_{t-\Delta t} \right]^D \right.$$

$$\left. - \Delta t \left(\delta_{t+\Delta t}^{A} - \delta_t^A + \left(\delta_{t-\Delta t} - \delta_t \right)^D \right) \right\}$$

Grouping arrival and departure point contributions:

$$q_{n+1}^{A} = \sum_{k=1}^{K} \Delta B_k \left\{ \frac{\Delta t}{2} Z(k)_t^A - \Delta t\, \delta_{t+\Delta t}^{A} + \Delta t\, \delta_t^A \right\}$$

$$+ \sum_{k=1}^{K} \Delta B_k \left\{ q_t + \Delta t \left[Z(k)_t - \frac{1}{2} Z(k)_{t-\Delta t} + \delta_t - \delta_{t-\Delta t} \right] \right\}^D$$

For the semi-implicit solution we write, noting that the vertical sum at arrival points collapses for δ_t^A and $\delta_{t+\Delta t}^A$:

$$Z = \Delta t\, \delta_t^A + \sum_{k=1}^{K} \frac{\Delta t}{2} \Delta B_k\, Z(k)_t^A +$$

$$\sum_{k=1}^{K} \Delta B_k \left\{ q_t + \Delta t \left[Z(k)_t - \frac{1}{2} Z(k)_{t-\Delta t} + \delta_t - \delta_{t-\Delta t} \right] \right\}^D$$

$$q_{t+\Delta t}^A \quad = \quad Z - \Delta t\, \delta_{t+\Delta t}^A \quad = \quad Z - \frac{\beta \Delta t}{2} \frac{1}{p_s^r} \sum_{j=1}^{K} \Delta p_j^r \left(D_j \right)_{t+\Delta t}^A$$

$$\frac{U_{t+\Delta t}^{A} - U_{t}^{D}}{\Delta t} = \left\{ fV - \frac{1}{a}\left[\frac{\partial \varphi}{\partial \lambda} + R_d T_{\mathrm{v}} \frac{\partial}{\partial \lambda} \ln p \right] \right\}_{t+\frac{\Delta t}{2}}^{M}$$

$$- \quad \frac{\beta}{2a} \Delta_{tt} \left[Y \frac{\partial T}{\partial \lambda} + R_d T_r \frac{\partial}{\partial \lambda} \ln p_s \right]$$

$$UN = fV - \frac{1}{a}\left[\frac{\partial \varphi}{\partial \lambda} + R_d T_{\mathrm{v}} \frac{\partial}{\partial \lambda} \ln p \right]$$

$$UL = \frac{1}{a}\left[Y \frac{\partial T}{\partial \lambda} + R_d T_r \frac{\partial \ln p_s}{\partial \lambda} \right]$$

$$\Delta_{tt} f = \left(f_{t+\Delta t}^{A} - 2 f_{t+\frac{\Delta t}{2}}^{M} + f_{t}^{D} \right)$$

$$\frac{U_{t+\Delta t}^{A} - U_{t}^{D}}{\Delta t} = UN_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2}\left[UL_{t+\Delta t}^{A} - 2UL_{t+\frac{\Delta t}{2}}^{M} + UL_{t}^{D} \right]$$

$$= (UN + \beta UL)_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2}\left[UL_{t+\Delta t}^{A} + UL_{t}^{D} \right] =$$

$$UNL = UN + \beta UL$$

$$= \quad (UNL)^{M}_{t+\frac{\Delta t}{2}} \quad - \quad \frac{\beta}{2}\left[UL^{A}_{t+\Delta t} + UL^{D}_{t}\right]$$

$$= \quad \frac{1}{2}\left[UNL^{A}_{t} + \left(2UNL_{t} - UNL_{t-\Delta t}\right)^{D}\right] \quad - \quad \frac{\beta}{2}UL^{A}_{t+\Delta t} \quad - \quad \frac{\beta}{2}UL^{D}_{t}$$

$$= \quad \frac{1}{2}UNL^{A}_{t} \quad + \quad \left[\left(UNL_{t} - \frac{1}{2}UNL_{t-\Delta t}\right) - \frac{\beta}{2}UL_{t}\right]^{D} \quad - \quad \frac{\beta}{2}UL^{A}_{t+\Delta t}$$

save UNL_{t} to be used in next step as $UNL_{t-\Delta t}$

$$U^{A}_{t+\Delta t} \quad + \quad \frac{\Delta t \beta}{2}UL^{A}_{t+\Delta t} \quad =$$

$$\frac{\Delta t}{2}UNL^{A}_{t} \quad + \quad \left(U_{t} + \Delta t\left(UNL_{t} - \frac{1}{2}UNL_{t-\Delta t} - \frac{\beta}{2}UL_{t}\right)\right)^{D} \quad = \quad \tilde{U}$$

$$\frac{V_{t+\Delta t}^{A} - V_{t}^{D}}{\Delta t} = \left\{ - fU - \frac{\cos(\theta)}{a} \left[\frac{\partial \varphi}{\partial \theta} + R_d T_{\mathrm{v}} \frac{\partial}{\partial \theta} \ln p \right] \right\}_{t+\frac{\Delta t}{2}}^{M}$$

$$- \frac{\beta \cos(\theta)}{2a} \Delta_{tt} \left[Y \frac{\partial T}{\partial \theta} + R_d T_r \frac{\partial}{\partial \theta} \ln p_s \right]$$

$$VN = - fU - \frac{\cos(\theta)}{a} \left[\frac{\partial \varphi}{\partial \theta} + R_d T_{\mathrm{v}} \frac{\partial}{\partial \theta} \ln p \right]$$

$$VL = \frac{\cos(\theta)}{a} \left[Y \frac{\partial T}{\partial \theta} + R_d T_r \frac{\partial \ln p_s}{\partial \theta} \right]$$

$$\Delta_{tt} f = \left(f_{t+\Delta t}^{A} - 2 f_{t+\frac{\Delta t}{2}}^{M} + f_{t}^{D} \right)$$

$$\frac{V_{t+\Delta t}^{A} - V_{t}^{D}}{\Delta t} = VN_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2} \left[VL_{t+\Delta t}^{A} - 2 VL_{t+\frac{\Delta t}{2}}^{M} + VL_{t}^{D} \right]$$

$$= (VN + \beta VL)_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2} \left[VL_{t+\Delta t}^{A} + VL_{t}^{D} \right] =$$

$$VNL = VN + \beta VL$$

$$= \quad (VNL)^{M}_{t+\frac{\Delta t}{2}} \quad - \quad \frac{\beta}{2}\left[VL^{A}_{t+\Delta t} + VL^{D}_{t}\right]$$

$$= \quad \frac{1}{2}\left[VNL^{A}_{t} + \left(2VNL_{t} - VNL_{t-\Delta t}\right)^{D}\right] \quad - \quad \frac{\beta}{2}VL^{A}_{t+\Delta t} \quad - \quad \frac{\beta}{2}VL^{D}_{t}$$

$$= \quad \frac{1}{2}VNL^{A}_{t} \quad + \quad \left[\left(VNL_{t} - \frac{1}{2}VNL_{t-\Delta t}\right) - \frac{\beta}{2}VL_{t}\right]^{D} \quad - \quad \frac{\beta}{2}VL^{A}_{t+\Delta t}$$

save VNL_{t} to be used in next step as $VNL_{t-\Delta t}$

$$V^{A}_{t+\Delta t} \quad + \quad \frac{\Delta t\beta}{2}VL^{A}_{t+\Delta t} \quad =$$

$$\frac{\Delta t}{2}VNL^{A}_{t} \quad + \quad \left(V_{t} + \Delta t\left(VNL_{t} - \frac{1}{2}VNL_{t-\Delta t} - \frac{\beta}{2}VL_{t}\right)\right)^{D} \quad = \quad \tilde{V}$$

Let $TN = \frac{\kappa T_{\mathrm{v}}\omega}{p}\left(\frac{1+\varepsilon q}{1+(\delta-1)q}\right), \quad TL = \tau D$

where τ is the matrix derived in the Eulerian model

The thermodynamic eq : $\frac{\mathrm{d}T_{\mathrm{v}}}{\mathrm{d}t} = TN - \frac{\beta}{2}\Delta_{tt}\tau D$

$$\frac{T_{t+\Delta t}^{A} - T_{t}^{D}}{\Delta t} = TN_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2}\left(TL_{t+\Delta t}^{A} - 2TL_{t+\frac{\Delta t}{2}}^{M} + TL_{t}^{D}\right)$$

$$= (TN + \beta TL)_{t+\frac{\Delta t}{2}}^{M} - \frac{\beta}{2}\left(TL_{t+\Delta t}^{A} + TL_{t}^{D}\right) =$$

Let $TNL = TN + \beta TL$ initialize $TNL(t-\Delta t)$

$$= \frac{1}{2}\left[TNL_{t}^{A} + \left(2TNL_{t} - TNL_{t-\Delta t}\right)^{D}\right] - \frac{\beta}{2}\left(TL_{t+\Delta t}^{A} + TL_{t}^{D}\right)$$

$$= \frac{1}{2}TNL_{t}^{A} - \frac{\beta}{2}TL_{t+\Delta t}^{A} + \left(TNL_{t} - \frac{1}{2}TNL_{t-\Delta t} - \frac{\beta}{2}TL_{t}\right)^{D}$$

$$T_{t+\Delta t}^{A} = -\frac{\beta\Delta t}{2}TL_{t+\Delta t}^{A} + Y$$

$$Y = \frac{\Delta t}{2}TNL_{t}^{A} + \left\{T_{t} + \Delta t\left(TNL_{t} - \frac{1}{2}TNL_{t-\Delta t} - \frac{\beta}{2}TL_{t}\right)\right\}^{D}$$

Semi-Implicit Semi-Lagrange

Momentum equations :

$$U^{A}_{t+\Delta t} = \tilde{U} - \frac{\beta \Delta t}{2} UL^{A}_{t+\Delta t}$$

$$V^{A}_{t+\Delta t} = \tilde{V} - \frac{\beta \Delta t}{2} VL^{A}_{t+\Delta t}$$

$$\overrightarrow{VL} = (UL, VL) = \cos\theta \, \nabla \left(AT + R_d \, T_r \, \ln p_s \right), \quad q = \ln p_s$$

rescaling to (u, v) and taking the divergence :

$$D^{+} = X - \frac{\beta \Delta t}{2} \nabla^{2} \left(AT^{+} + R_d \, T_r \, q^{+} \right)$$

$$\text{where} \qquad X = \nabla \cdot \left[\frac{1}{\cos\theta} \left(\tilde{U}, \tilde{V} \right) \right]$$

Thermodynamic and surface pressure equations :

$$T^{A}_{t+\Delta t} = Y - \frac{\beta \Delta t}{2} \tau \, D^{A}_{t+\Delta t}$$

$$q^{A}_{t+\Delta t} = Z - \Delta t \, S \cdot D^{A}_{t+\Delta t}$$

$$\text{let} \qquad F^{+} = F^{A}_{t+\Delta t}\,, \quad b = \frac{\beta \Delta t}{2}\,, \quad r = R_{d}\, T_{r}$$

$$D^{+} \quad = \quad X - b\,\nabla^{2}\left[A\left(Y - b\,\tau\,D^{+}\right) \quad + \quad r\left(Z - \Delta t\; S \cdot D^{+}\right)\right]$$

$$\left[I + b\,\nabla^{2}\left(-bA\tau - r\;\Delta t\;S\right)\right]D^{+} \quad = \quad X - b\,\nabla^{2}\left(AY + rZ\right)$$

$$\left[I + \frac{n(n+1)}{a^{2}}\,b\left(bA\tau + r\;\Delta t\;S\right)\right]D^{+}$$

$$SI(1)$$

$$= \quad X + \frac{n(n+1)}{a^{2}}\,b\left(AY + rz\right)$$

$$\text{let} \qquad D\_m \quad =$$

$$I + \frac{n(n+1)}{a^{2}}\left[\left(\frac{\beta\Delta t}{2}\right)^{2} A\tau + \left(\frac{\beta\Delta t}{2}\right)^{2}\frac{R_{d}}{P^{r}}\,T_{r}\cdot\Delta p^{r}\right]$$

$$D\_m \quad = \quad I + \frac{n(n+1)}{a^{2}}\,\frac{\beta^{2}(\Delta t)^{2}}{4}\left[A\tau + \frac{R_{d}}{P^{r}}\,T_{r}\cdot\Delta p^{r}\right]$$

and :

$$T_r \cdot \Delta P^r \;=\; \begin{pmatrix} T_1^r \\ \vdots \\ T_{Levs}^r \end{pmatrix} \begin{pmatrix} \mathit{\Delta} p_1^r & \cdots & \mathit{\Delta} p_{Levs}^r \end{pmatrix}$$

Since $\quad S \;=\; \dfrac{\beta}{2P_s^r} \begin{pmatrix} \mathit{\Delta} p_1^r & \cdots & \mathit{\Delta} p_K^r \end{pmatrix}$

We can use subroutine get_cd_hyb with $\dfrac{\mathit{\Delta} t}{2}$ to get D_m for slg.

This result is identical to the Eulerian forward step formulation.

Eq. SI(1) can now be solved for D^+.

$$D^+ = D_m^{-1}\left[x + \frac{n(n+1)}{a^2}\frac{\beta \Delta t}{2}\left(AY + R_d T_r Z \right) \right]$$

Finally T^+ and q^+ can also be isolated.

mass adjustment for slg

$$\tilde{D}^{+} = X + \tilde{X} - \frac{\beta}{2}\Delta t \nabla^{2}\left(A\tilde{T}^{+} + RT_{r}\,\tilde{q}^{+}\right)$$

$$D^{+} = X \qquad - \frac{\beta}{2}\Delta t \nabla^{2}\left(AT^{+} + RT_{r}\,\tilde{q}^{+}\right)$$

subtract

$$\tilde{D}^{+} - D^{+} = \tilde{X} - \frac{\beta}{2}\Delta t \nabla^{2}\left\{A\left(\tilde{T}^{+} - T^{+}\right) + RT_{r}\left(\tilde{q}^{+} - q^{+}\right)\right\}$$

$$\tilde{T}^{+} = Y + \tilde{Y} - \frac{\beta \Delta t}{2}\tau \tilde{D}^{+}$$

$$T^{+} = Y \qquad - \frac{\beta \Delta t}{2}\tau D^{+}$$

subtract

$$\tilde{T}^{+} - T^{+} = \tilde{Y} - \frac{\beta \Delta t}{2}\tau\left(\tilde{D}^{+} - D^{+}\right)$$

$$\tilde{q}^{+} = Z + \tilde{Z} - \frac{\beta}{2}\Delta t S \tilde{D}^{+}$$

$$q^{+} = Z - \frac{\beta \Delta t}{2} S \cdot D^{+} \qquad S = \frac{1}{P_s^r}\left(\Delta P_1^r \ \ldots \ \Delta P_{Levs}^r \right)$$

subtract

$$\tilde{q}^{+} - q^{+} = \tilde{Z} - \frac{\beta}{2}\Delta t S \cdot \left(\tilde{D}^{+} - D^{+} \right)$$

let $D^{*} = \tilde{D}^{+} - D^{+}$, $T^{*} = \tilde{T}^{+} - T^{+}$, $q^{*} = \tilde{q}^{+} - q^{+}$

then

$$D^{*} = \tilde{X} - \frac{\beta}{2}\Delta t \nabla^{2}\left(AT^{*} + RT_r q^{*} \right)$$

$$T^{*} = \tilde{Y} - \frac{\beta}{2}\Delta t \tau D^{*}$$

$$q^{*} = \tilde{Z} - \frac{\beta}{2}\Delta t S \cdot D^{*}$$

These equations have the same form as the semi-implicit system for the solution of the dynamic part.

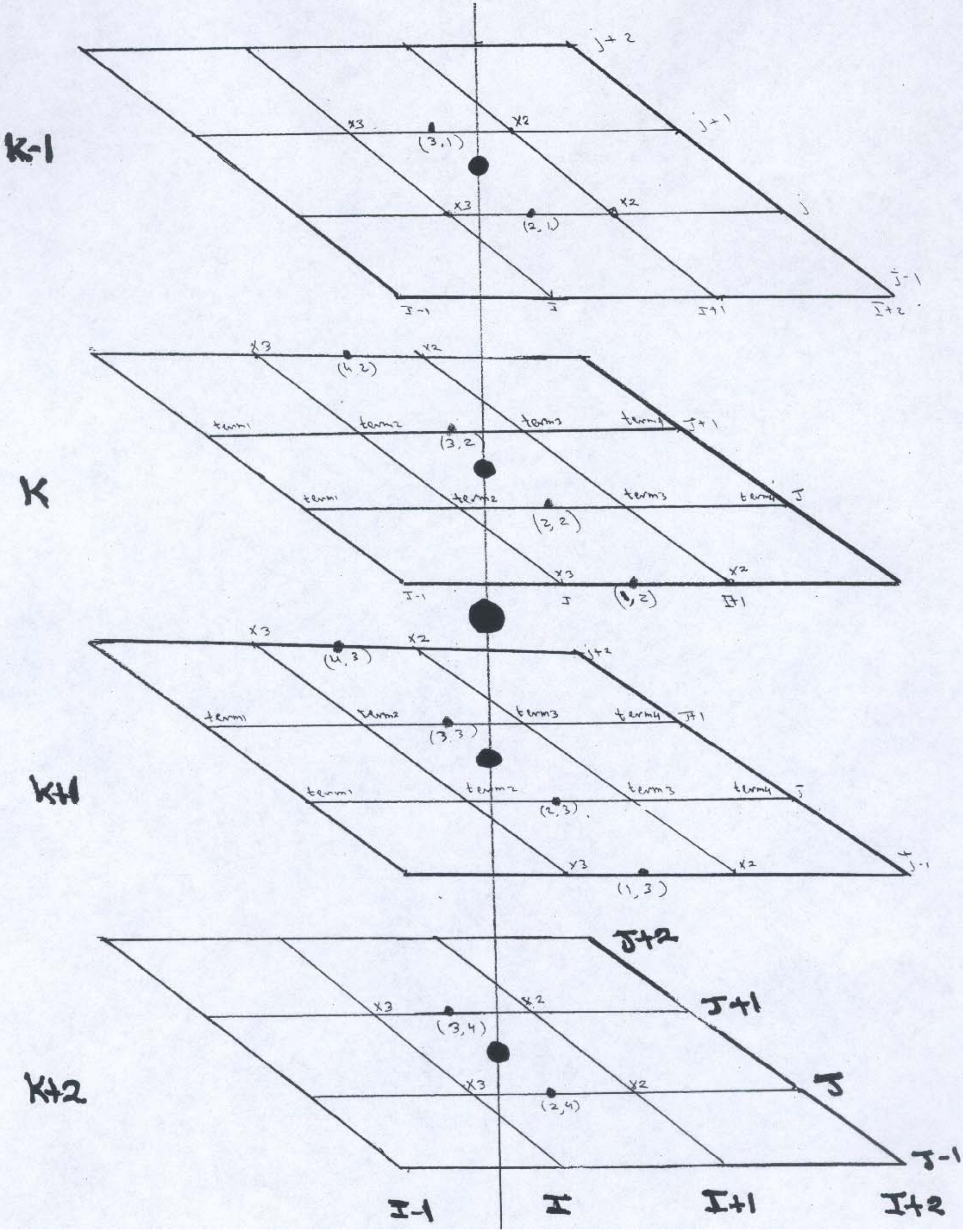
$$D^* = \tilde{X} - \frac{\beta}{2}\Delta t \nabla^2 \left[A\left(\tilde{Y} - \frac{\beta}{2}\Delta t \tau D^* \right) + RT_r \left(\tilde{Z} - \frac{\beta}{2}\Delta t S \cdot D^* \right) \right]$$

$$\left[I - \left(\frac{\beta}{2}\Delta t \right)^2 \nabla^2 \{ A\tau + RT_r S \cdot D \} \right] D^* = \tilde{X} - \beta \frac{\Delta t}{2} \nabla^2 \left[A\tilde{Y} + RT_r \tilde{Z} \right]$$

$$D^* = D_m^{-1} \left\{ \tilde{X} + \frac{n(n+1)}{a^2} \left(\beta \frac{\Delta t}{2} \right) \left[A\tilde{Y} + RT_r \tilde{Z} \right] \right\}$$

$$\tilde{D}^+ = D^* + D^+$$

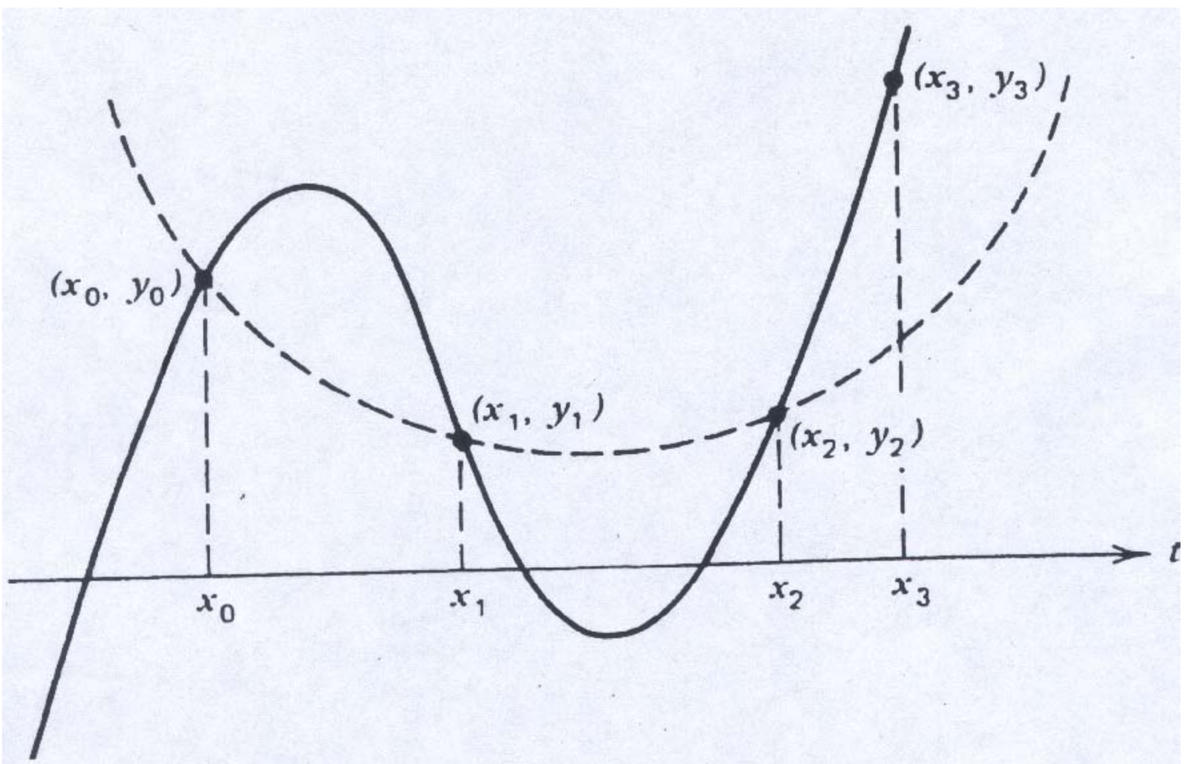
$$\tilde{q}^+ = q^* + q^+$$


K-1
K
K+1
K+2
I-1
I
I+1
I+2
J+2
J+1
J
J-1
x3
x2
term1
term2
term3
term4
(3,1)
(2,1)
(4,2)
(3,2)
(2,2)
(1,2)
(4,3)
(3,3)
(2,3)
(1,3)
(3,4)
(2,4)

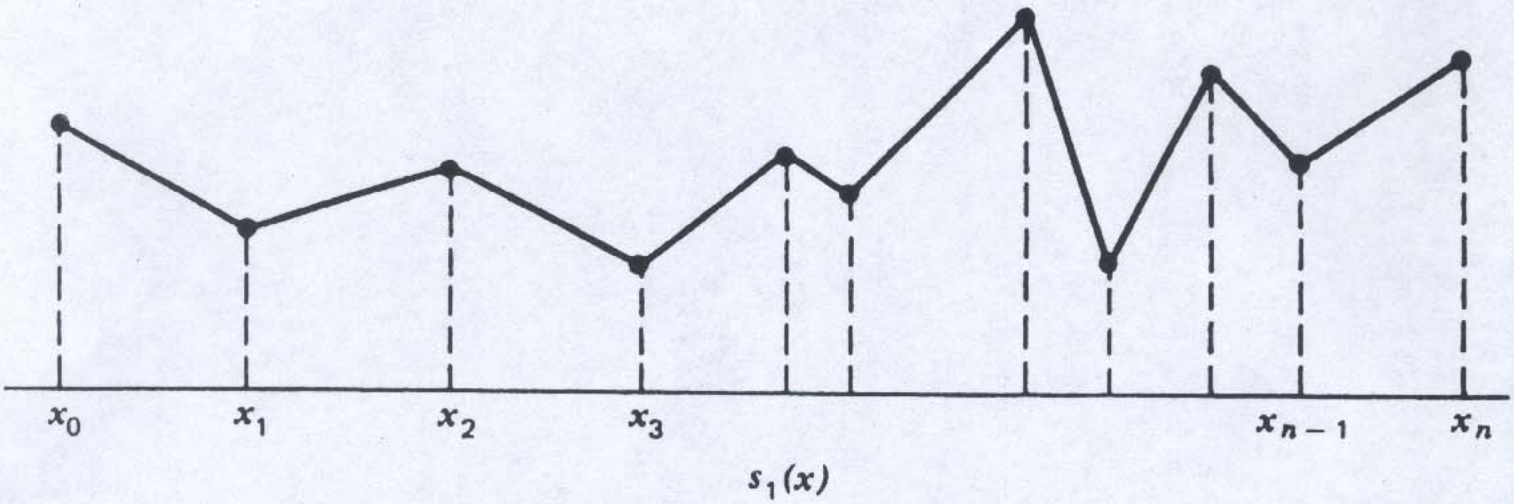
$$\boxed{p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x),}$$

$$l_i(x) = \frac{w(x)}{(x - x_i)\, w'(x_i)}$$

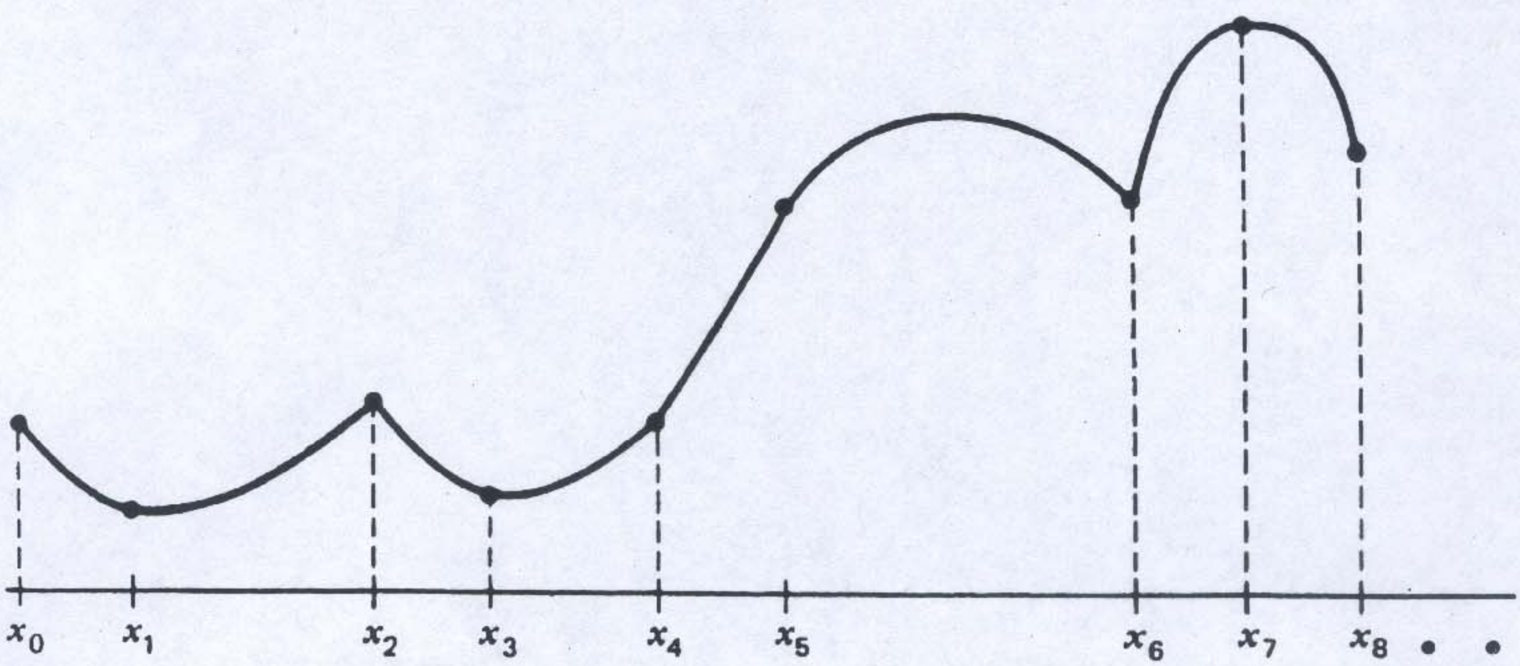
$$= \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}.$$



(x_3, y_3)
(x_0, y_0)
(x_1, y_1)
(x_2, y_2)
t
x_0
x_1
x_2
x_3



A piecewise linear Lagrange polynomial.



A piecewise quadratic Lagrange polynomial.

$$p(a) = f(a), \qquad p'(a) = f'(a),$$

$$p(b) = f(b), \qquad p'(b) = f'(b)$$

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

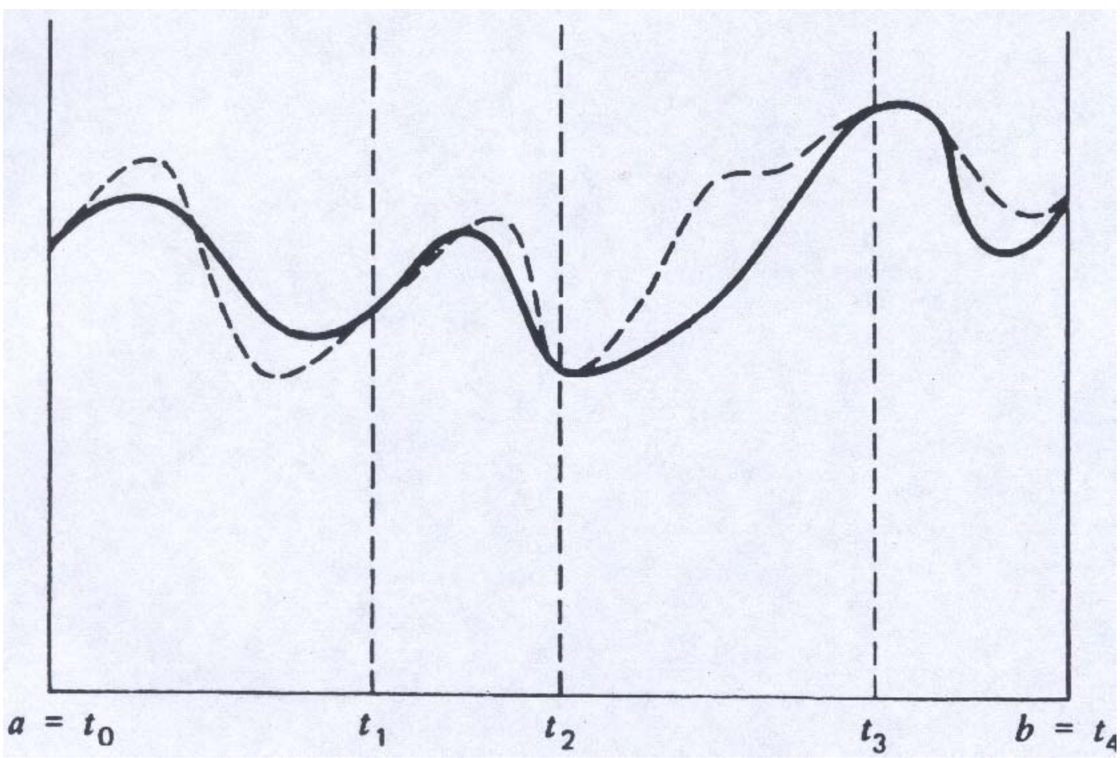
$$p(t) = p(a)\phi_1(t) + p(b)\phi_2(t) + p'(a)\psi_1(t) + p'(b)\psi_2(t),$$

$$\phi_1(t) = \frac{(t-b)^2[(a-b) + 2(a-t)]}{(a-b)^3}$$

$$\phi_2(t) = \frac{(t-a)^2[(b-a) + 2(b-t)]}{(a-b)^3}$$

$$\psi_1(t) = \frac{(t-a)(t-b)^2}{(a-b)^2}$$

$$\psi_2(t) = \frac{(t-a)^2(t-b)}{(a-b)^2}$$



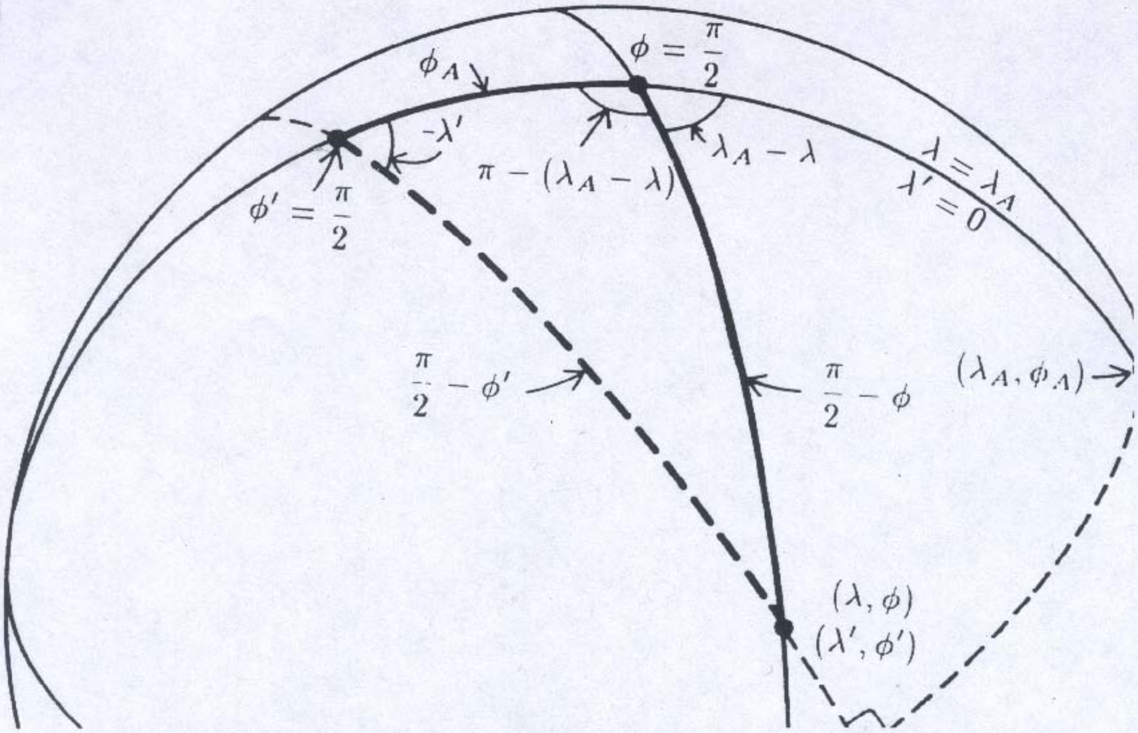
$a = t_0$
t_1
t_2
t_3
$b = t_4$

$$\lambda_D + \int_{(\lambda_D,\phi_D,t)}^{(\lambda_A,\phi_A,t+\Delta t)} \frac{u(\lambda,\phi,t)}{\cos\phi}\,dt = \lambda_A$$

$$\phi_D + \int_{(\lambda_D,\phi_D,t)}^{(\lambda_A,\phi_A,t+\Delta t)} v(\lambda,\phi,t)\,dt = \phi_A$$

$$\lambda_D{}^{k+1} = \lambda_A - \frac{1}{2}\Delta t\left[\frac{u(\lambda_A,\phi_A,t+\Delta t)}{\cos\phi_A} + \frac{u(\lambda_D{}^k,\phi_D{}^k,t)}{\cos\phi_D{}^k}\right]$$

$$\phi_D{}^{k+1} = \phi_A - \frac{1}{2}\Delta t[v(\lambda_A,\phi_A,t+\Delta t) + v(\lambda_D{}^k,\phi_D{}^k,t)]$$



$$\sin\phi' = \sin\phi \cos\phi_A - \cos\phi \sin\phi_A \cos(\lambda_A - \lambda)$$

$$\sin\lambda' \cos\phi' = -\sin(\lambda_A - \lambda) \cos\phi.$$

$$\lambda'_D = -\Delta t \left[\frac{u'(\lambda'_*, \phi'_*, t + \Delta t/2)}{\cos\phi'_*} \right]$$

$$\phi'_D = -\Delta t \left[v'\left(\lambda'_*, \phi'_*, t + \frac{\Delta t}{2} \right) \right]$$

$$\lambda'^{k+1}_* = -\frac{\Delta t}{2} \left[\frac{u'(\lambda'^k_*, \phi'^k_*, t + \Delta t/2)}{\cos\phi'^k_*} \right]$$

$$\phi'^{k+1}_* = -\frac{\Delta t}{2} \left[v'\left(\lambda'^k_*, \phi'^k_*, t + \frac{\Delta t}{2} \right) \right]$$

Performance

		T382k64	T510k64
Euler		1.	
Lagrange	la_qb	3.	
Lagrange	la_lb	3.75	1.95
Hermite	la_qb	2.87	
Hermite	la_lb	3.46	2.04