Lagrangian Data Assimilation and Observing System Design from Dynamical Systems Perspective

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Project Sponsored by
Lagrangian Properties of the Ocean

- Coherent structures
  - Commonly observed
  - Descriptive physical phenomena are often in Lagrangian nature

- Lagrangian trajectories
  - Directly related to dynamics
  \[ dx = v(x, t) dt + d\eta \]
  - Maybe associated with Lagrangian structures
Combining the Elements

Data Assimilation Using Eulerian Models

Lagrangian Observations Along the Paths

Lagrangian Data Assimilation

Dynamical Systems Theory

Observing System Design: Optimal Deployment Strategies
Eulerian Model and Observation

**Eulerian model**

\[
\begin{align*}
\mathbf{x}_F^I(t_k) &= \mathbf{m}_F(\mathbf{x}_F^a, t_{k-1}) \\
\mathbf{P}_F^I(t_k) &= \mathcal{M}_F(\mathbf{P}_F^a, t_{k-1})
\end{align*}
\]

**Eulerian observation at** \( \mathbf{r}_s \)

\[
\begin{align*}
\mathbf{x}_F^a &= \mathbf{x}_F^I + \mathbf{K}_F(\mathbf{y}_S^o - \mathbf{H}_S \mathbf{x}_F^I) \\
\mathbf{K}_F &= \mathbf{P}_F^I \mathbf{H}_S^T(\mathbf{H}_S \mathbf{P}_F^I \mathbf{H}_S^T + \mathbf{R}_S^o)^{-1} \\
\mathbf{P}_F^I &= \mathcal{E}\left([\mathbf{x}_F^I - \mathbf{x}_F^I](\mathbf{x}_F^I - \mathbf{x}_F^I)^T\right) \\
\mathbf{P}_F^a &= \mathcal{A}\left(\mathbf{P}_F^I, \mathbf{H}_S, \mathbf{R}_S^o\right) = (\mathbf{I} - \mathbf{K}_F \mathbf{H}_S)\mathbf{P}_F^I
\end{align*}
\]
Lagrangian Observations and Eulerian Model

- Observation $y^o_k$ of the true positions $y^t_k$ subject to noise $\varepsilon^t_k$ along the instrument path, $k=1,\ldots,K$,

$$y^o_k = y^t_k + \varepsilon^t_k$$

$$\varepsilon^t_k \sim N(0, R^t_k)$$

- True drifter dynamics (may be stochastic)

$$dy^t = v(y^t, t)dt + d\eta$$

$$y^t(t_k) = y^t(t_0) + \int_{t_0}^{t_k} v(y^t, t)dt + d\eta$$

- Model forecast $x_D(t_k)$

- Simulated drifter dynamics (deterministic)

$$dx^t_D = H_v x^t_F(x^t_D, t)dt$$

$$x^t_D(t_k) = x^t_D(t_{k-1}) + \int_{t_{k-1}}^{t_k} H_v x^t_F(x^t_D, t)dt$$

Observing System Design for Lagrangian Data Assimilation  NOAA EMC Seminar  June 30, 2005  K. Ide
Lagrangian Data Assimilation (LaDA)

- Augmented system for the ocean ($x_F$) and drifter ($x_D$) states
  - State vector $x$ is $N=N_F + N_D$ dimensional & dynamics is one-way interaction
  \[
  x = \begin{pmatrix} x_F \\ x_D \end{pmatrix}, \quad \frac{d}{dt} \left( \begin{pmatrix} x_F \\ x_D \end{pmatrix} \right) = \begin{pmatrix} m_F(x_F, t) \\ m_D(x_F, x_D, t) \end{pmatrix}, \quad \bar{x} = E \begin{pmatrix} x_F \\ x_D \end{pmatrix}
  \]

- Error covariance $P$ is $N \times N = (N_F + N_D)(N_F + N_D)$ dimensional
  \[
  P = \begin{pmatrix} P_{FF} & P_{FD} \\ P_{DF} & P_{DD} \end{pmatrix} = E \begin{bmatrix} \Delta x_F (\Delta x_F)^T & \Delta x_F (\Delta x_D)^T \\ \Delta x_D (\Delta x_F)^T & \Delta x_D (\Delta x_D)^T \end{bmatrix}, \quad \Delta x = \begin{pmatrix} \Delta x_F \\ \Delta x_D \end{pmatrix} = \begin{pmatrix} x_F - \bar{x}_F \\ x_D - \bar{x}_D \end{pmatrix}
  \]

- Update mechanism
  \[
  \begin{pmatrix} x_F^a \\ x_D^a \end{pmatrix} = \begin{pmatrix} x_F^f \\ x_D^f \end{pmatrix} + \begin{pmatrix} K_{FD}^f \\ K_{DD}^f \end{pmatrix} \left( y_D^o - H_D \begin{pmatrix} x_F^f \\ x_D^f \end{pmatrix} \right) = \begin{pmatrix} x_F^f \\ x_D^f \end{pmatrix} + \begin{pmatrix} P_{FD}^f (P_{DD}^f + R_{DD}^o)^{-1} \\ P_{DD}^f (P_{DD}^f + R_{DD}^o)^{-1} \end{pmatrix} (y_D^o - x_D^f)
  \]
  - Lagrangian observation operator $H_D = (0 \ I)$ extracts the drifter information
  \[
  H_D x = x_D, \quad H_D P = \begin{pmatrix} P_{FD}^f \\ P_{DD}^f \end{pmatrix}
  \]
  - Correlation $P_{FD}$ enables to propagate $(y_D^o - x_D^f)$ into $x_F^a$ as well as $x_D^f$
Ensemble Kalman Filter (EnKF)

\[ x^f_j(t_k) = m(x^a_j, t_{k-1}) \]

\[ y_k^o = H_k x^f_k(t_k) + \epsilon_k^o \]

Analysis

\[ x^a_j(t_k) = x^f_j(t_k) + K_k (y^o_j - H_k x^f_j) \]

\[ K_k = P^f(t_k)(H_k)^T \left( H_k P^f(t_k)(H_k)^T + R_k^o \right)^{-1} \]

\[ P^f(t_k)(H_k)^T \approx \frac{1}{N_e - 1} \sum_{j=1}^{N_e} \left[ x^f_j(t_k) - \bar{x}^f(t_k) \right] \left[ H_k x^f_j(t_k) - H_k \bar{x}^f(t_k) \right]^T \]

\[ H_k P^f(t_k)(H_k)^T \approx \frac{1}{N_e - 1} \sum_{j=1}^{N_e} \left[ H_k x^f_j(t_k) - H_k \bar{x}^f(t_k) \right] \left[ H_k x^f_j(t_k) - H_k \bar{x}^f(t_k) \right]^T \]

\[ \bar{x}^f(t_k) = \frac{1}{N_e} \sum_{j=1}^{N_e} x^f_j(t_k) \]
Application to Shallow-Water Ocean Circulation

Can LaDA recover $x^t(t)$ after assimilating the drifter positions $y^o(t_k)$?

- How many drifters?
- How often?
- Where to deploy?

T=0 (IC)

control run
- $h_{ave}=500m$

80 Ensemble members
- $h_{ave}=550m$
- $h_{std}=50m$
One Drifter at $v = 500 \text{m}^2\text{s}^{-1}$ & $(\Delta T, N_e, r_{\text{loc}}) = (1\text{day}, 80, \infty)$

- Truth (Control run)
- With DA
- Without DA

T=0 (IC)

T=90 days
Performance Verification

Parameters

- Degree of turbulence $\nu$
- Assimilation time interval $\Delta T$
- Ensemble number $N_e$
- Localization length scale $r_{loc}$

Performance validation by comparing

- True error norm
  \[
  |h|^t = \sqrt{\sum_{i,j}^{N_x,N_y} \left( \bar{h}_{i,j}^t - h_{i,j}^t \right)^2 / \sum_{i,j}^{N_x,N_y} \left( \bar{h}_{i,j}^t \right)^2}
  \]

- Predicted error and ensemble spread
  \[
  |KE|^t = \sqrt{\sum_{i,j}^{N_x,N_y} \left( \bar{u}_{i,j}^t - u_{i,j}^t \right)^2 + \left( \bar{v}_{i,j}^t - v_{i,j}^t \right)^2 / \sum_{i,j}^{N_x,N_y} \left( \bar{u}_{i,j}^t \right)^2 + \left( \bar{v}_{i,j}^t \right)^2}
  \]

- True error norm
  \[
  |h|^t = \sqrt{\sum_{i,j,k}^{N_x,N_y,N_e} \left( \bar{h}_{i,j,k}^t - h_{i,j,k}^t \right)^2 / N_e \sum_{i,j}^{N_x,N_y} \left( \bar{h}_{i,j}^t \right)^2}
  \]

- Predicted error and ensemble spread
  \[
  |KE|^t = \sqrt{\sum_{i,j,k}^{N_x,N_y,N_e} \left( \bar{u}_{i,j,k}^t - u_{i,j,k}^t \right)^2 + \left( \bar{v}_{i,j,k}^t - v_{i,j,k}^t \right)^2 / N_e \sum_{i,j}^{N_x,N_y} \left( \bar{u}_{i,j}^t \right)^2 + \left( \bar{v}_{i,j}^t \right)^2}
  \]
Lagrangian Decorrelation Time Scale ($T_L$) at $\nu = 500$ m$^2$s$^{-1}$

Region A: $T_L \sim O(10 \text{ days})$

Region B: $T_L \sim O(100+ \text{ days})$

Results for $T_L$ are similar at $\nu = 400$ m$^2$s$^{-1}$
Effect of Assimilation Time Interval $\Delta T$

- Lagrangian time scale is about $T_L \sim 10$ days
  - $\Delta T = 1, 2, 4, 5, 8, 10, 15$ and $20$ days
  - $(\nu, r_{loc}, N_e) = (500 \text{m}^2\text{s}^{-1}, 300 \text{km}, 80)$
- The method is stable if $\Delta T < T_L$
Effect of Turbulence $\nu$

- Reducing $\nu$ leads to turbulent flow
  - $\nu=500\text{m}^2\text{s}^{-1}$ to $400\text{m}^2\text{s}^{-1}$
  - $\nu_{\text{loc}}=(\infty, 1\text{day}, 80)$
- Convergence deteriorates relative to $\nu=500\text{m}^2\text{s}^{-1}$
- Predicted error does not match true error
- Increasing to 36 drifters does not rectify the problem

Norm comparison

Chaotic advection of $x_D$
Effect of Ensemble Size $N_e$ and Localization $r_{loc}$

- Error covariance is approximated by ensembles;
  - Slow convergence $\sim (N_e)^{-1/2}$
  - Noisy correlation between remote regions for small $N_e$
  leading to deterioration of filtering

- A remedy is to introduce localization of error correlation
  
  $$ K = \rho \circ (PH^T) (\rho \circ (HPH^T) + R)^{-1} $$

  - $\rho \circ ()$ is the function of distance between
    - Grid points
    - Grid and drifter
    - drifters
  denoting the Schur product (Hamill, 2001)

  - $r_{loc}$ gives the radius of influence
Effect of Localization $r_{loc}$ for Turbulent Dynamics

Three localization radii investigated

- $r_{loc} = 150\text{km}, 300\text{km}, 600\text{m}$
- $(\nu, \Delta T, N_e) = (400\text{m}^2\text{s}^{-1}, 1\text{days}, 80)$ and 36 drifters

Better convergence using the localization

- Optimal $r_{loc} = 300\text{km}$

![Norm comparison](image1)

![Chaotic advection of $x_D$](image2)
Drifter Update Examples

- Along the jet
  - Large derivation: can be still successful

- In the recirculating region

- In the eddies
  - Detrainment process by the saddle

Can LaDA handle chaotic drifter dynamics?
Dealing with Lagrangian Saddle: Estimation of Coherent Structures

Parameters \((\sigma, \rho, \Delta T) = (0.04, 0.02, 1.5)\)

Ex: failure case for I.C.no.3
Saddle Effect with the Eulerian Model

- Large updates in drifter position occur near saddle point
  - Prior PDF distribution can be bimodal; Unimodal Gaussian distribution breaks down.
  - It can potentially produce large and spurious changes in the flow

Ensemble Spread of drifter

Update mechanism of the mean
Saddle Effect

- Saddle effect occurs near the linearly hyperbolic region of velocity
  (λ: hyperbolicity given by the positive local Lyapunov exponent)

\[
\begin{align*}
\frac{d}{dt} x_D &= M_D (x_F) x_D \\
\approx x_D(t) &= F_D (x_F) x_D(t_0) \\
\frac{d}{dt} P_{DD} &= M_D (x_F) P_{DD} + P_{DD} M_D (x_F)^T \\
&\approx P_{DD}(t) = F_D (x_F) P_{DD}(t_0) F_D (x_F)^T
\end{align*}
\]

- \(\Delta x^a_D\) can be updated correctly: \(P_{DD}\) grows linearly in time with exponent 2λ:
  \[
  \Delta x^a_D = P_{DD}^f \left( P_{DD}^f + R_D^o \right)^{-1} \left( y_D^o - x_D^f \right)
  \]

- \(\Delta x^a_F\) may be unreasonably large because
  \[
  \Delta x^a_F = P_{FD}^f \left( P_{FD}^f + R_D^o \right)^{-1} \left( y_D^o - x_D^f \right)
  \]

- \(P_{FD}\) may be approximated by too large with exponent λ:
- Dragged by a large innovation \((y_D^o - x_D^o)\)
Tracer Control: Sanity check for $\Delta x^a_F$

$C_F \equiv$ Standard deviation of $\Delta x^a_F$ with respect to the expected error

- $P^f_{FF}$ is independent of $x^f_D$ or $x^t_D$: the flow has no knowledge of the drifter.

$$C_F = (\Delta x^a_F)^T (P^f_{FF})^{-1} \Delta x^a_F$$

$$\Delta x^a_F = P^f_{FD} (P^f_{DD} + R^o_D)^{-1} (y^o_D - x^f_D)$$

**Implementation**

- if $C_F < \delta$: Update of $x_F$
- if $C_F \geq \delta$: No update of $x_F$ (but $x_D$ is updated)

$C_F$ is computed for $x_D$ within $r_{FD}$ from $x_D$ \hspace{2cm} ($r_{FD} < r_{loc}$)

Control is applied to entire $x_D$ within $r_{loc}$ from $x_D$
Tracer Control with Vortex System

$\Delta T = 1.5$ and $\delta = 3$

- **EKF**
  - Without TC
  - With TC

- **EnKF**
  - Without TC
  - With TC
Dynamical Systems Theory

- Two-dimensional drifter dynamics
  \[ \frac{dx}{dt} = u(x, y, t) = -\frac{\partial}{\partial y} \psi(x, y, t) \times a(x, y, t) \]
  \[ \frac{dy}{dt} = v(x, y, t) = \frac{\partial}{\partial x} \psi(x, y, t) \times a(x, y, t) \]
  
  - Velocity \( u(x, y, t) \), is tangent to the streamfunction \( \psi(x, y, t) \)

- Steady flow
  \[ \frac{d}{dt} \psi(x_D(t), y_D(t)) = 0 \]
  \[ \psi(x_D(t), y_D(t)) = \psi(x_D(t_0), y_D(t_0)) \]
  
  - Any trajectory remains on the iso- \( \psi(x, y) \) curve
  - Streamfunction field \( \psi(x, y) \) completely describes the glot
  - Stable and unstable trajectories from the hyperbolic fixed point (saddle) define the global template

- Unsteady flow
  
  - Stable and unstable invariant manifolds (material curve) from the hyperbolic trajectory (Lagrangian saddle) define the global template of the Lagrangian dynamics
Targeted Observing System Design

- Centers (eddies)
- Hyperbolic trajectories
- Mixed cases
  - Centers
  - Hyperbolic trajectories
Preliminary Results

\[ |h| = \sqrt{\sum_{i,j} \left( \bar{h}_{i,j} - h'_{i,j} \right)^2 / \sum_{i,j} (h'_{i,j})^2} \]

\[ |KE| = \sqrt{\sum_{i,j} \left( \bar{u}_{i,j} - u'_{i,j} \right)^2 + \left( \bar{v}_{i,j} - v'_{i,j} \right)^2 / \sum_{i,j} (u'_{i,j})^2 + (v'_{i,j})^2} \]
Targeted Observation System

- Control

- Centers

- Mixed

- Hyperbolic trajectories

T=365 days
Observing System Design for Transient Flow Dynamics: Finite Time Lyapunov Exponents
Future Directions: Atmospheric Applications
Atmospheric motion vector / Cloud wind vector

- Global wind vectors are used for the initial value of Numerical Weather Prediction (NWP), via data assimilation.

- Satellite-based observations provide wide coverage for the monitoring of atmospheric motion vectors by tracking the movement of clouds and interpolating the movement in time.

Idea: Use Lagrangian data assimilation method & assimilate the cloud feature positions directly into the morel, - without any interpolation in time - cloud height information is naturally taken care of
Summary and Work in Progress

Data Assimilation Using Eulerian Models

Lagrangian Observations Along the Paths

Lagrangian Data Assimilation

Dynamical Systems Theory

Observing System Design: Optimal Deployment Strategies