BASIC ENERGETICS OF MULTI-POINT SHALLOW-WATER EQUATIONS

$$\boxed{\begin{aligned}\frac{\partial h}{\partial t} &= -\mathbf{M}\{\nabla\cdot(h\mathbf{v})\}\\ \frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{N}\left\{\nabla\left(\mathbf{v}^2/2+gh\right)\right\}\end{aligned}}$$

$\mathbf{M}, \mathbf{N}$: Linear, weighted horizontal-average operators that satisfy

$$\mathbf{M}\{\nabla\cdot\mathbf{A}\} = \nabla\cdot\left[\mathbf{M}\{\mathbf{A}\}\right] \text{ and } \mathbf{N}\{\nabla\cdot\mathbf{A}\} = \nabla\cdot\left[\mathbf{N}\{\mathbf{A}\}\right]$$

$$\overline{\mathbf{M}\{\mathbf{A}\}\cdot\mathbf{B}} = \overline{\mathbf{A}\cdot\mathbf{M}\{\mathbf{B}\}} \quad \text{and} \quad \overline{\mathbf{N}\{\mathbf{A}\}\cdot\mathbf{B}} = \overline{\mathbf{A}\cdot\mathbf{N}\{\mathbf{B}\}}$$

$\overline{(\ )}$: average over the entire domain

Then

$$\frac{\partial}{\partial t}\left[\overline{h\left(\mathbf{v}^2+gh\right)/2}\right] = \overline{h\mathbf{v}\cdot\left[\mathbf{M}\left\{\nabla(\mathbf{v}^2/2+gh)\right\} - \mathbf{N}\left\{\nabla\left(\mathbf{v}^2/2+gh\right)\right\}\right]}$$

Energy is conserved if $\mathbf{M} = \mathbf{N}$.