Lagrangian Data Assimilation and Observing System Design from Dynamical Systems Perspective

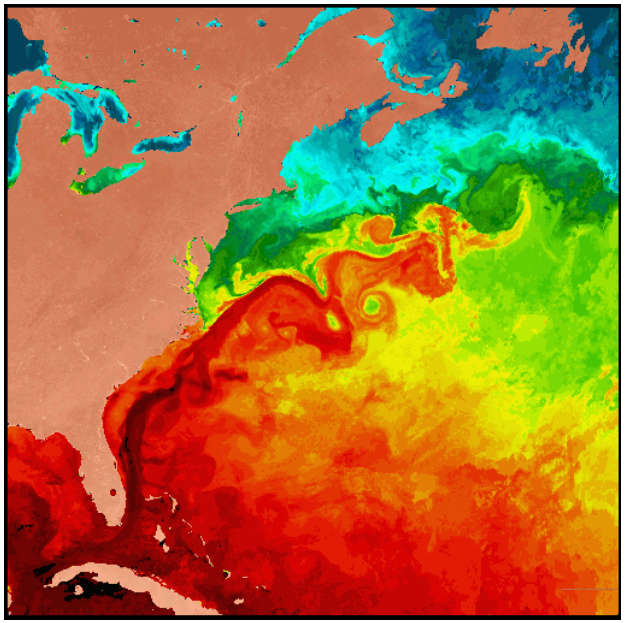
Kayo Ide, UCLA

Chris Jones, Hayder Salman and Liyan Liu, UNC-CH

Project Sponsored by

Lagrangian Properties of the Ocean

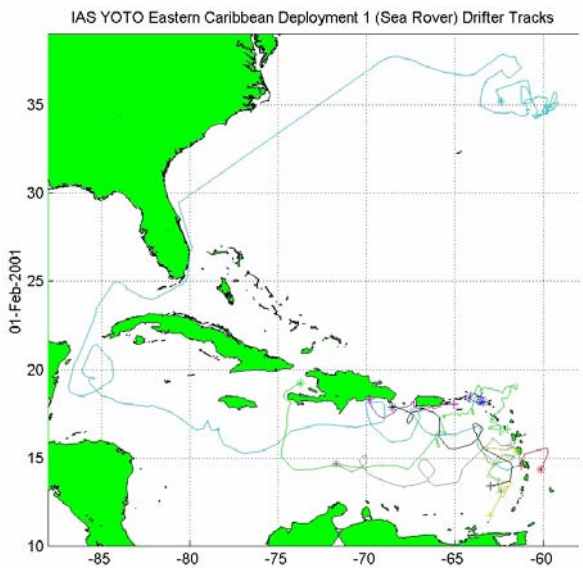
- Coherent structures
  - Commonly observed
  - Descriptive physical pheonomena are often in Lagrangian nature



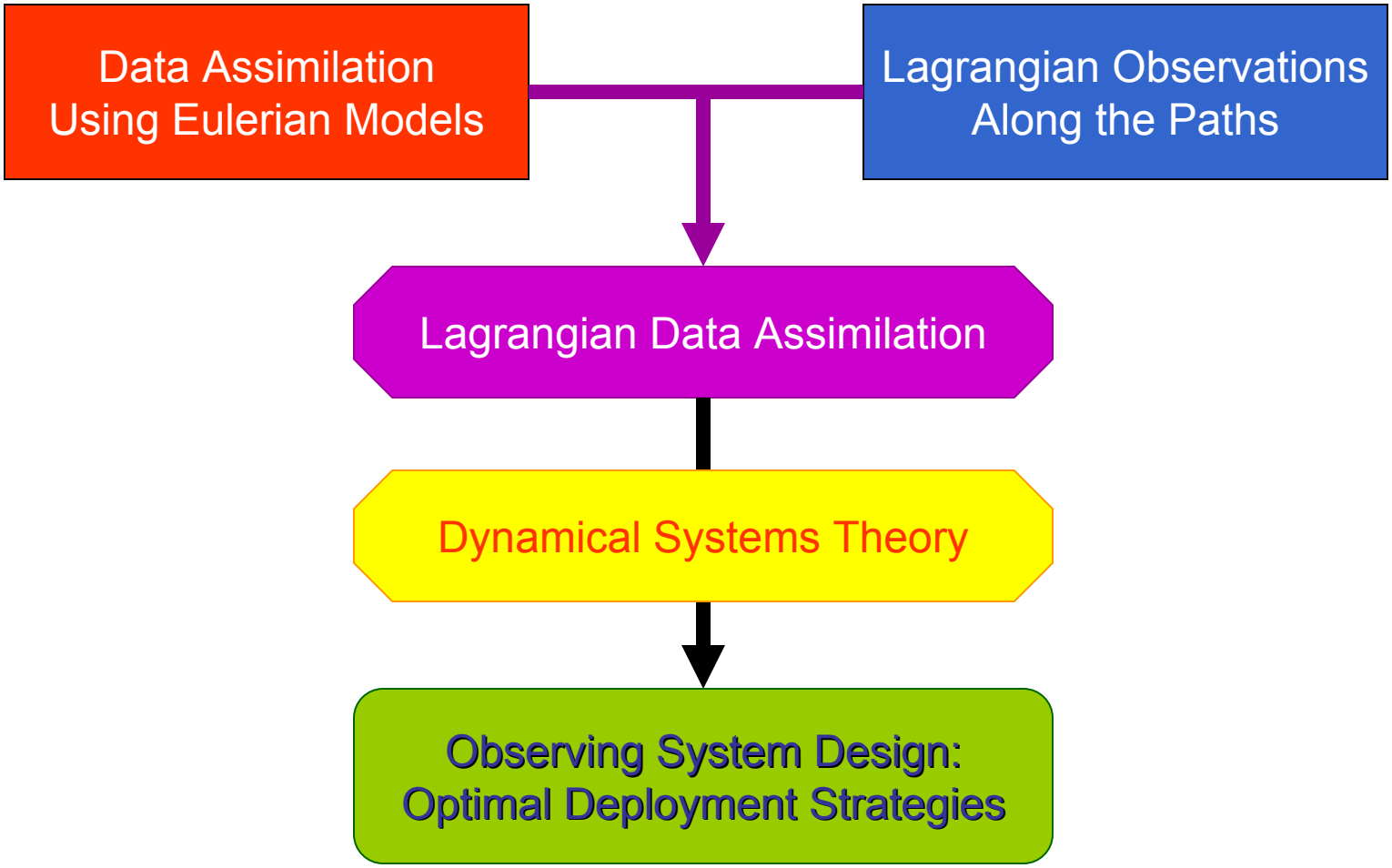
- Lagrangian trajectories
  - Directly related to dynamics

$$d\mathbf{x} = \mathbf{v}(\mathbf{x}, t)dt + d\boldsymbol{\eta}$$

  - Maybe associated with Lagrangian structures

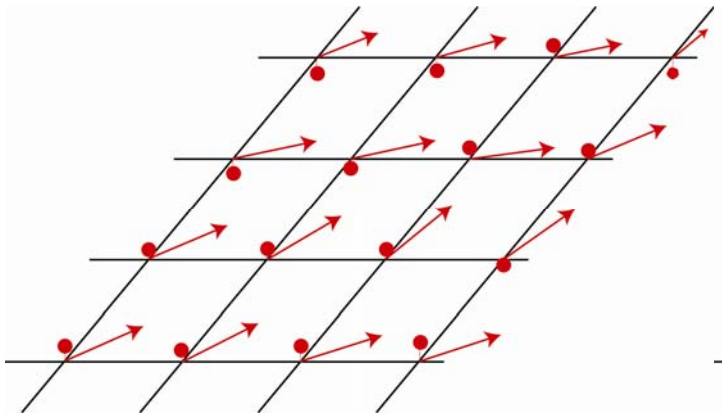


Combining the Elements

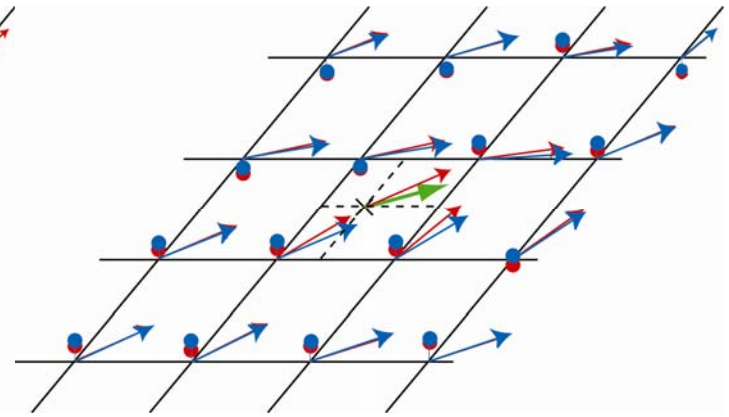


Eulerian Model and Observation

- Eulerian model



- Eulerian observation at $\mathbf{r}_s$



Forecast

$$\mathbf{x}_F^f(t_k) = \mathbf{m}_F(\mathbf{x}_F^a \quad t_{k-1})$$

$$\mathbf{P}_F^f(t_k) = \mathcal{M}_F(\mathbf{P}_F^a \quad t_{k-1})$$

Analysis

$$\mathbf{x}_F^a = \mathbf{x}_F^f + \mathbf{K}_F(\mathbf{y}_S^o - \mathbf{H}_{Sv}\mathbf{x}_F^f)$$

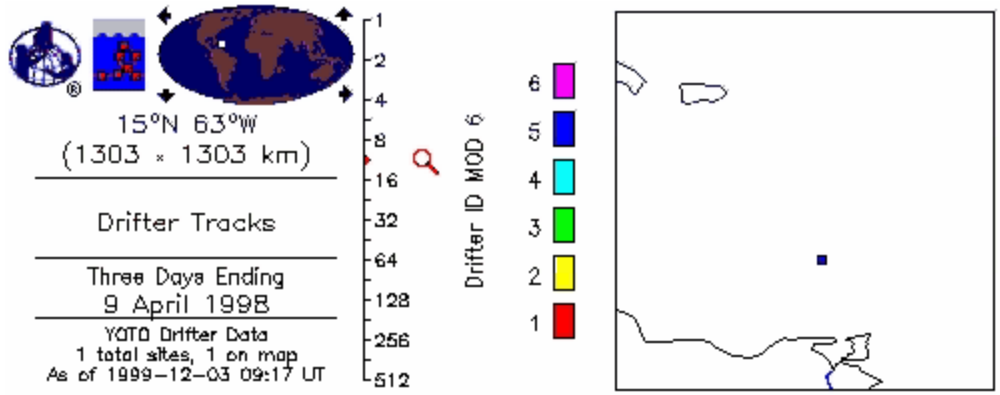
$$\mathbf{K}_F = \mathbf{P}_{FF}^f \mathbf{H}_{Sv}^T (\mathbf{H}_{Sv}\mathbf{P}_F^f\mathbf{H}_{Sv}^T + \mathbf{R}_S^o)^{-1}$$

$$\mathbf{P}_F^f = E\left[(\mathbf{x}_F^f - \mathbf{x}_F^t)(\mathbf{x}_F^f - \mathbf{x}_F^t)^T\right]$$

$$\mathbf{P}_F^a = \mathcal{A}(\mathbf{P}_F^f \quad \mathbf{H}_{Sv} \quad \mathbf{R}_S^o) = (\mathbf{I} - \mathbf{K}_F\mathbf{H}_{Sv})\mathbf{P}_F^f$$

Kalman Filter

Lagrangian Observations and Eulerian Model



- Observation $\mathbf{y}^{o}_{k}$ of the true positions $\mathbf{y}^{t}_{k}$ subject to noise $\boldsymbol{\varepsilon}^{t}_{k}$ along the instrument path, k=1,…,K,

$$\mathbf{y}^{o}_{k} = \mathbf{y}^{t}_{k} + \boldsymbol{\varepsilon}^{t}_{k} \qquad \boldsymbol{\varepsilon}^{t}_{k} \sim N\left(0 \quad \mathbf{R}^{t}_{k}\right)$$

  - True drifter dynamics (may be stochastic)

$$d\mathbf{y}^{t} = \mathbf{v}\left(\mathbf{y}^{t} \quad t\right)dt + d\boldsymbol{\eta} \qquad d\boldsymbol{\eta}: \text{dynamic noise}$$

$$\mathbf{y}^{t}\left(t_{k}\right) = \mathbf{y}^{t}\left(t_{0}\right) + \int_{t_0}^{t_k} \mathbf{v}\left(\mathbf{y}^{t} \quad t\right)dt + d\boldsymbol{\eta}$$

- Model forecast $\mathbf{x}_{D}(t_{k})$
  - Simulated drifter dynamics (deterministic)

$$d\mathbf{x}^{f}_{D} = \mathbf{H}_{v}\mathbf{x}^{f}_{F}\left(\mathbf{x}^{f}_{D} \quad t\right)dt$$

$$\mathbf{x}^{f}_{D}\left(t_{k}\right) = \mathbf{x}^{f}_{D}\left(t_{k-1}\right) + \int_{t_{k-1}}^{t_k} \mathbf{H}_{v}\mathbf{x}^{f}_{F}\left(\mathbf{x}^{f}_{D} \quad t\right)dt$$

Lagrangian Data Assimilation (LaDA)

- Augmented systemfor the ocean ($\mathbf{x}_F$) and drifter ($\mathbf{x}_D$) states
  - State vector $\mathbf{x}$ is $N=N_F + N_D$ dimensional & dynamics is one-way interaction

$$\mathbf{x} \equiv \begin{pmatrix} \mathbf{x}_F \\ \mathbf{x}_D \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} \mathbf{x}_F \\ \mathbf{x}_D \end{pmatrix} = \begin{pmatrix} \mathbf{m}_F(\mathbf{x}_F \quad t) \\ \mathbf{m}_D(\mathbf{x}_F \quad \mathbf{x}_D \quad t) \end{pmatrix} \qquad \bar{\mathbf{x}} = E\begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_D \end{bmatrix}$$

  - Error covariance $\mathbf{P}$ is $N\mathrm{x}N=(N_F + N_D)(N_F + N_D)$ dimensional

$$\mathbf{P} \equiv \begin{pmatrix} \mathbf{P}_{FF} & \mathbf{P}_{FD} \\ \mathbf{P}_{DF} & \mathbf{P}_{DD} \end{pmatrix} = E\begin{bmatrix} \Delta\mathbf{x}_F(\Delta\mathbf{x}_F)^T & \Delta\mathbf{x}_F(\Delta\mathbf{x}_D)^T \\ \Delta\mathbf{x}_D(\Delta\mathbf{x}_F)^T & \Delta\mathbf{x}_D(\Delta\mathbf{x}_D)^T \end{bmatrix} \qquad \Delta\mathbf{x} \equiv \begin{pmatrix} \Delta\mathbf{x}_F \\ \Delta\mathbf{x}_D \end{pmatrix} = \begin{pmatrix} \mathbf{x}_F - \bar{\mathbf{x}}_F \\ \mathbf{x}_D - \bar{\mathbf{x}}_D \end{pmatrix}$$

- Update mechanism

$$\begin{pmatrix} \mathbf{x}_F^a \\ \mathbf{x}_D^a \end{pmatrix} = \begin{pmatrix} \mathbf{x}_F^f \\ \mathbf{x}_D^f \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{FD}^f \\ \mathbf{K}_{DD}^f \end{pmatrix}\left[\mathbf{y}_D^o - \mathbf{H}_D\begin{pmatrix} \mathbf{x}_F^f \\ \mathbf{x}_D^f \end{pmatrix}\right] = \begin{pmatrix} \mathbf{x}_F^f \\ \mathbf{x}_D^f \end{pmatrix} + \begin{pmatrix} \mathbf{P}_{FD}^f\left(\mathbf{P}_{DD}^f + \mathbf{R}_{DD}^o\right)^{-1} \\ \mathbf{P}_{DD}^f\left(\mathbf{P}_{DD}^f + \mathbf{R}_{DD}^o\right)^{-1} \end{pmatrix}\left(\mathbf{y}_D^o - \mathbf{x}_D^f\right)$$

  - Lagrangian observation operator $\mathbf{H}_D = (\mathbf{0}\ \mathbf{I})$ extracts the drifter information

$$\mathbf{H}_D\mathbf{x} = \mathbf{x}_D \qquad \mathbf{H}_D\mathbf{P} = \begin{pmatrix} \mathbf{P}_{FD} \\ \mathbf{P}_{DD} \end{pmatrix}$$

  - Correlation $\mathbf{P}_{FD}$ enables to propagate ($\mathbf{y}^o_D - \mathbf{x}^f_D$) into $\mathbf{x}^a_F$ as well as $\mathbf{x}^f_D$

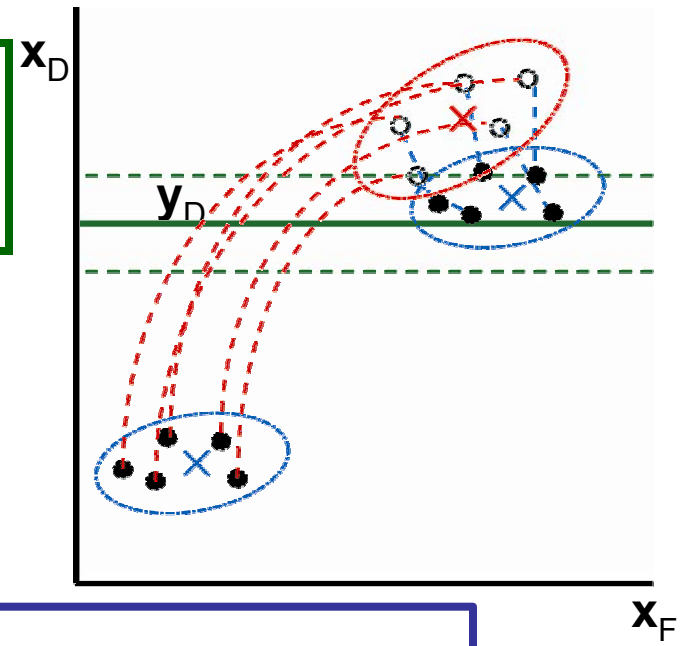
Ensemble Kalman Filter (EnKF)

Ensemble Forecast

$$\mathbf{x}_j^{\mathrm{f}}(t_k) = \mathbf{m}\left(\mathbf{x}_j^{\mathrm{a}} \quad t_{k-1}\right)$$

Observation

$$\mathbf{y}_k^{\mathrm{o}} = \mathbf{H}_k \mathbf{x}_k^{\mathrm{t}}(t_k) + \boldsymbol{\varepsilon}_k^{\mathrm{o}}$$



Analysis

$$\mathbf{x}_j^{\mathrm{a}}(t_k) = \mathbf{x}_j^{\mathrm{f}}(t_k) + \mathbf{K}_k\left(\mathbf{y}_j^{\mathrm{o}} - \mathbf{H}_k \mathbf{x}_j^{\mathrm{f}}\right)$$

$$\mathbf{K}_k = \mathbf{P}^{\mathrm{f}}(t_k)(\mathbf{H}_k)^T\left(\mathbf{H}_k \mathbf{P}^{\mathrm{f}}(t_k)(\mathbf{H}_k)^T + \mathbf{R}_k^{\mathrm{o}}\right)^{-1}$$

$$\mathbf{P}^{\mathrm{f}}(t_k)(\mathbf{H}_k)^T \approx \frac{1}{N_e - 1}\sum_{j=1}^{N_e}\left[\mathbf{x}_j^{\mathrm{f}}(t_k) - \overline{\mathbf{x}}^{\mathrm{f}}(t_k)\right]\left[\mathbf{H}_k \mathbf{x}_j^{\mathrm{f}}(t_k) - \mathbf{H}_k \overline{\mathbf{x}}^{\mathrm{f}}(t_k)\right]^T$$

$$\mathbf{H}_k \mathbf{P}^{\mathrm{f}}(t_k)(\mathbf{H}_k)^T \approx \frac{1}{N_e - 1}\sum_{j=1}^{N_e}\left[\mathbf{H}_k \mathbf{x}_j^{\mathrm{f}}(t_k) - \mathbf{H}_k \overline{\mathbf{x}}^{\mathrm{f}}(t_k)\right]\left[\mathbf{H}_k \mathbf{x}_j^{\mathrm{f}}(t_k) - \mathbf{H}_k \overline{\mathbf{x}}^{\mathrm{f}}(t_k)\right]^T$$

$$\overline{\mathbf{x}}^{\mathrm{f}}(t_k) = \frac{1}{N_e}\sum_{j=1}^{N_e}\mathbf{x}_j^{\mathrm{f}}(t_k)$$

Application to Shallow-Water Ocean Circulation

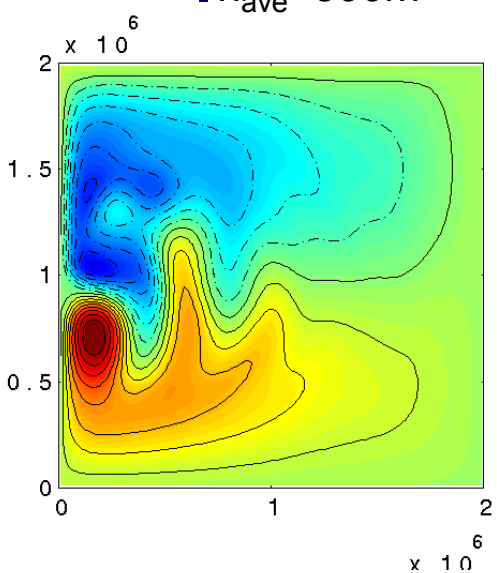
Can LaDA recover $\mathbf{x}^t(t)$ after assimilating the drifter positions $\mathbf{y}^o(t_k)$?

- How many drifters?
- How often?
- Where to deploy?

T=0
(IC)

control run

h_{ave}=500m

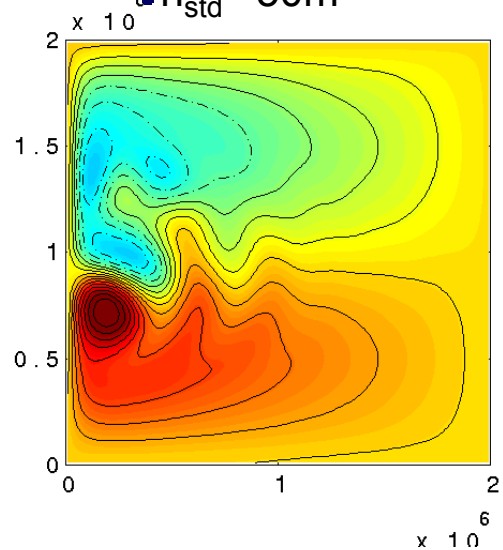


?

80 Ensemble members

h_{ave}=550m

h_{std} =50m

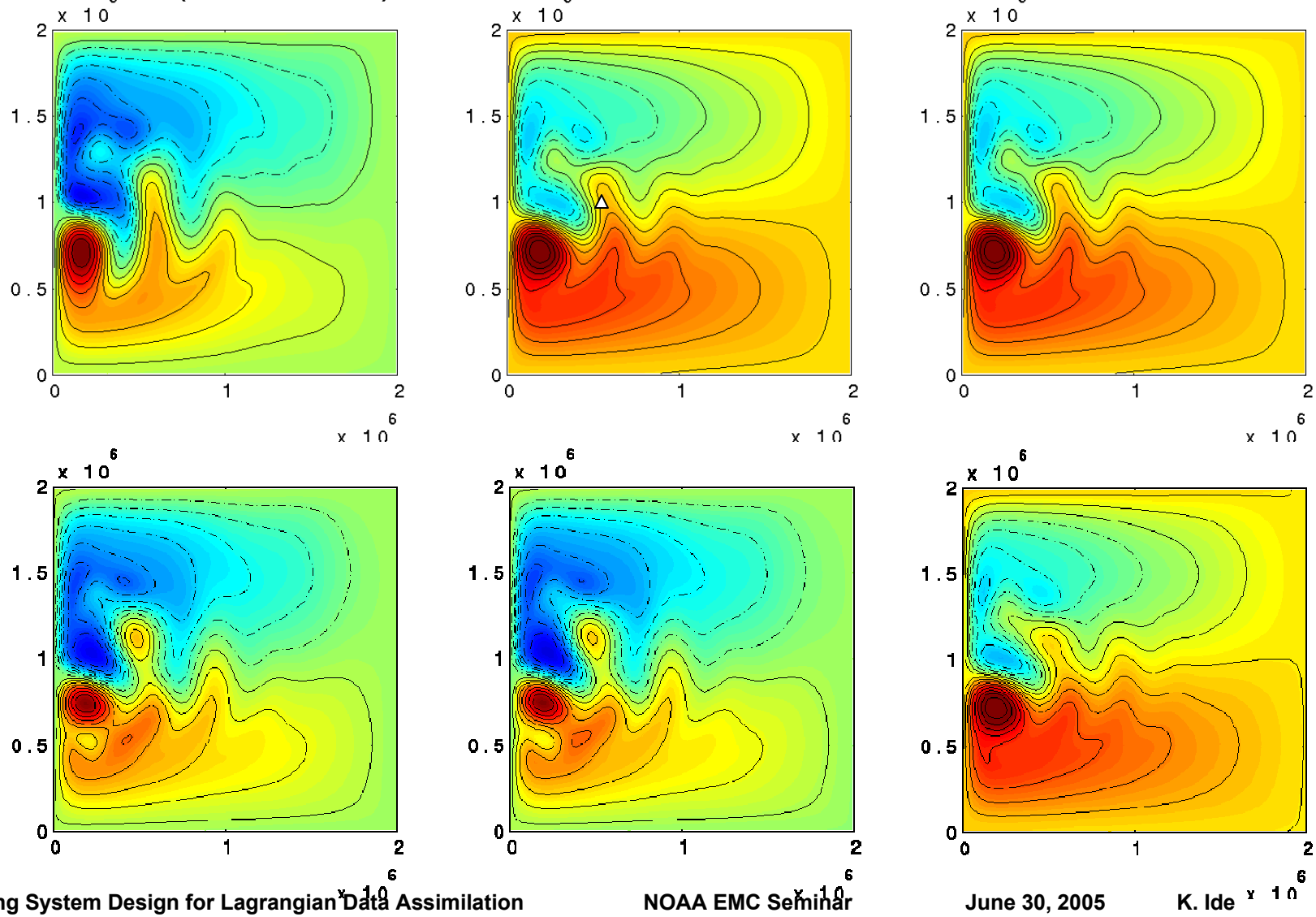


One Drifter at v= 500m^{2}s^{-1} & (ΔT, N_e, r_{loc}) =(1day, 80, ∞)

- Truth (Control run)
- With DA
- Without DA

T=0 (IC)

T=90 days



Performance Verification

- Parameters
  - Degree of turbulence v
  - Assimilation time interval ΔT
  - Ensemble number N_e
  - Localization length scale r_{loc}

- Performance validation by comparing
  - True error norm

$$|h|^{t} = \sqrt{\sum_{i,j}^{Nx,Ny} \left(\bar{h}^{f}_{i,j} - h^{t}_{i,j}\right)^2 / \sum_{i,j}^{Nx,Ny} \left(\bar{h}^{t}_{i,j}\right)^2}$$

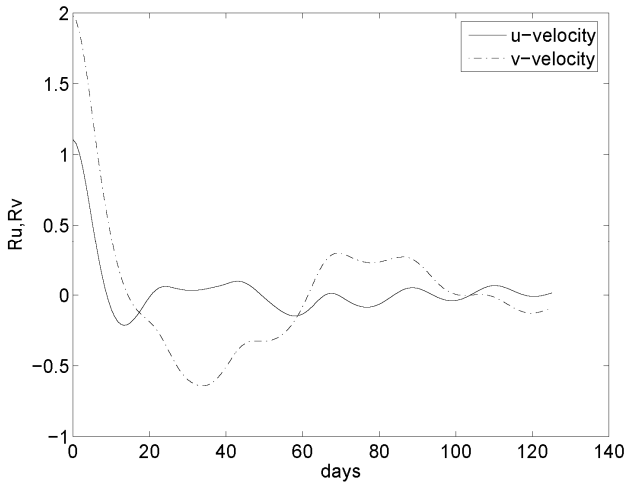
$$|KE|^{t} = \sqrt{\sum_{i,j}^{Nx,Ny} \left(\bar{u}^{f}_{i,j} - u^{t}_{i,j}\right)^2 + \left(\bar{v}^{f}_{i,j} - v^{t}_{i,j}\right)^2 / \sum_{i,j}^{Nx,Ny} \left(\bar{u}^{t}_{i,j}\right)^2 + \left(\bar{v}^{t}_{i,j}\right)^2}$$

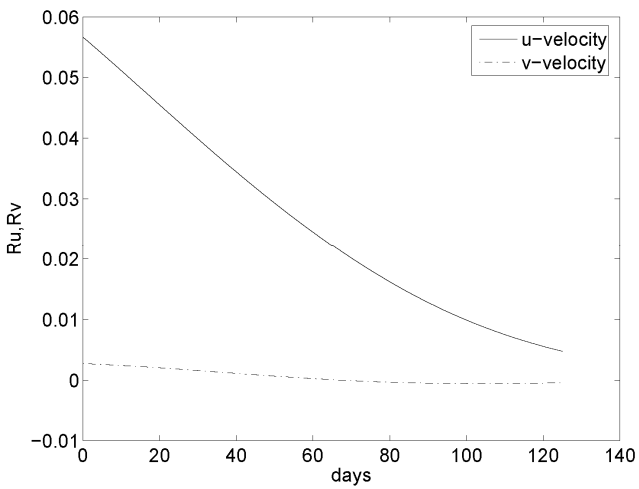
  - Predicted error and ensemble spread

$$|h|^{f} = \sqrt{\sum_{i,j,k}^{Nx,Ny,N_e} \left(\bar{h}^{f}_{i,j} - h^{f}_{i,j,k}\right)^2 / N_e \sum_{i,j}^{Nx,Ny} \left(\bar{h}^{f}_{i,j}\right)^2}$$

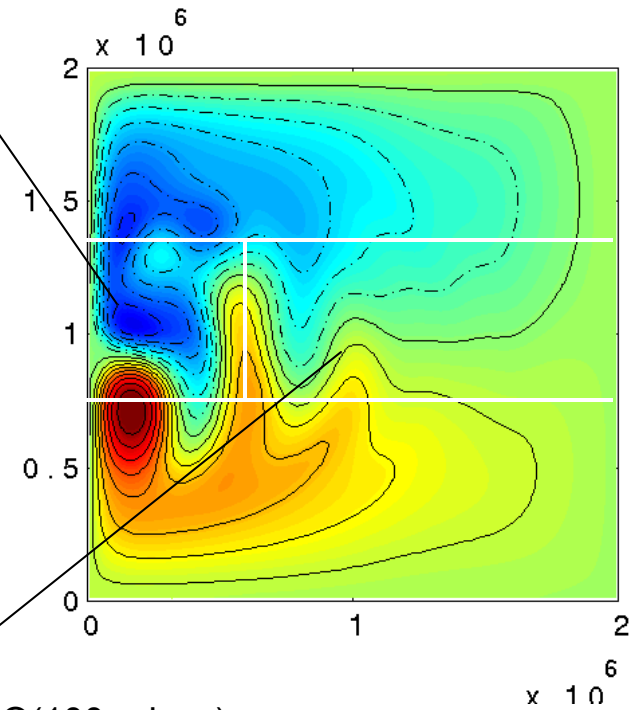
$$|KE|^{f} = \sqrt{\sum_{i,j,k}^{Nx,Ny,N_e} \left(\bar{u}^{f}_{i,j} - u^{f}_{i,j,k}\right)^2 + \left(\bar{v}^{f}_{i,j} - v^{f}_{i,j,k}\right)^2 / N_e \sum_{i,j}^{Nx,Ny} \left(\bar{u}^{f}_{i,j}\right)^2 + \left(\bar{v}^{f}_{i,j}\right)^2}$$

Lagrangian Decorrelation Time Scale (T_L) at v= 500m^{2}s^{-1}





Region A: T_L ~ O(10 days)



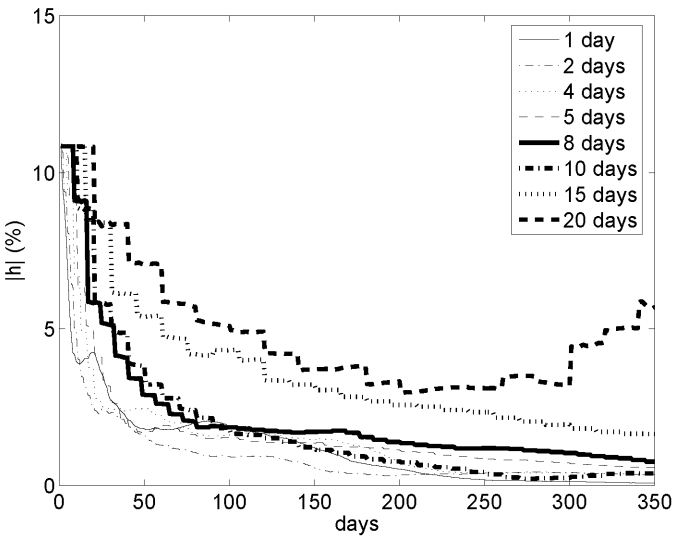
Region B: T_L ~ O(100+ days)

- Results for T_L are similar at v= 400m^{2}s^{-1}

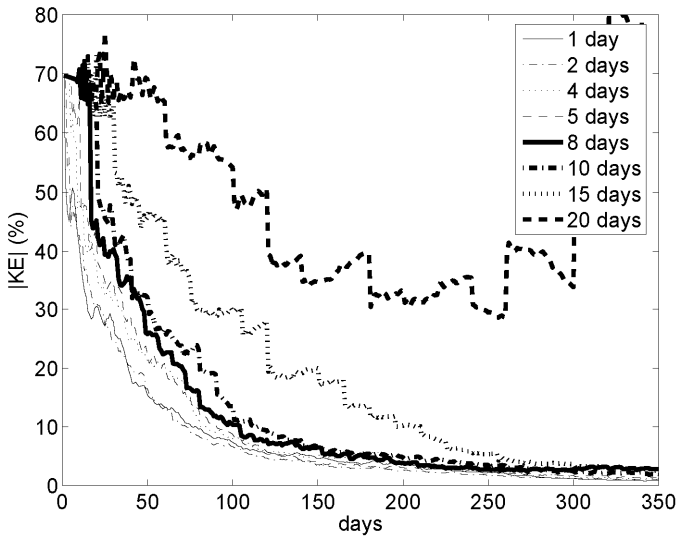
Effect of Assimilation Time Interval ΔT

- Lagrangian time scale is about T_L~10days
  - ΔT =1,2,4,5,8,10,15 and 20 days
  - (v, r_{loc}, N_e) =(500m^{2}s^{-1}, 300km, 80)
- The method is stable if $\Delta T < T_L$

Height Error

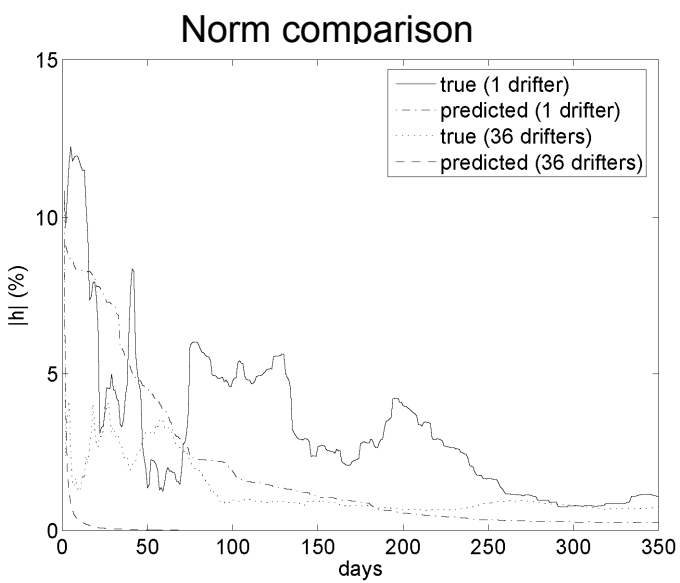


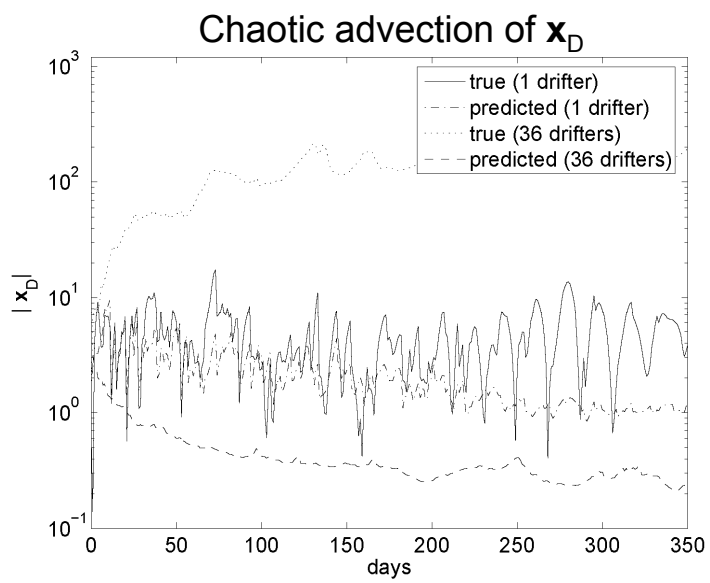
Kinetic Energy Error



Effect of Turbulence ν

- Reducing ν leads to turbulent flow
  - ν=500m^{2}s^{-1} to 400m^{2}s^{-1}
  - (r_{loc} , ΔT, N_e) =(∞, 1day, 80)
- Convergence deteriorate relative to ν=500m^{2}s^{-1}
- Predicted error does not match true error
- Increasing to 36 drifters does not rectify the problem





Effect of Ensemble Size N_e and Localization r_{loc}

- Error covariance is approximated by ensembles;
  - Slow convergence ~ $(N_e)^{-1/2}$
  - Noisy correlation between remote regions for small N_e

  leading to deterioration of filtering

- A remedy is to introduce localization of error correlation

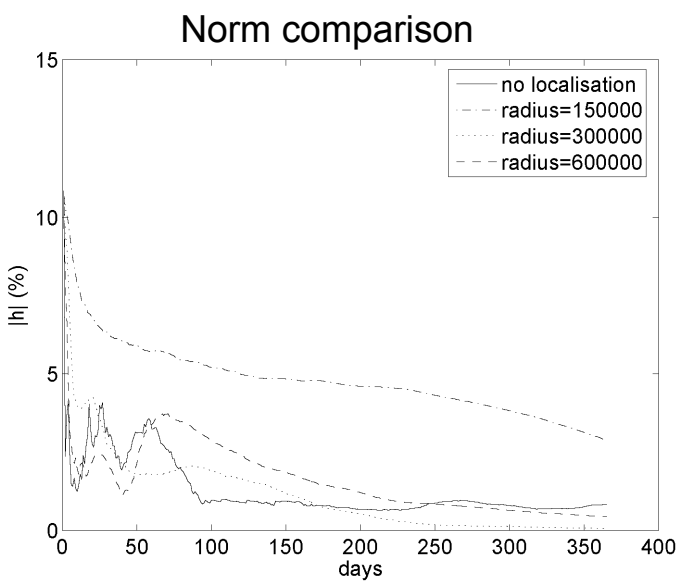
$$\mathbf{K} = \rho \circ (\mathbf{PH}^T) \, (\rho \circ (\mathbf{HPH}^T) + \mathbf{R})^{-1}$$

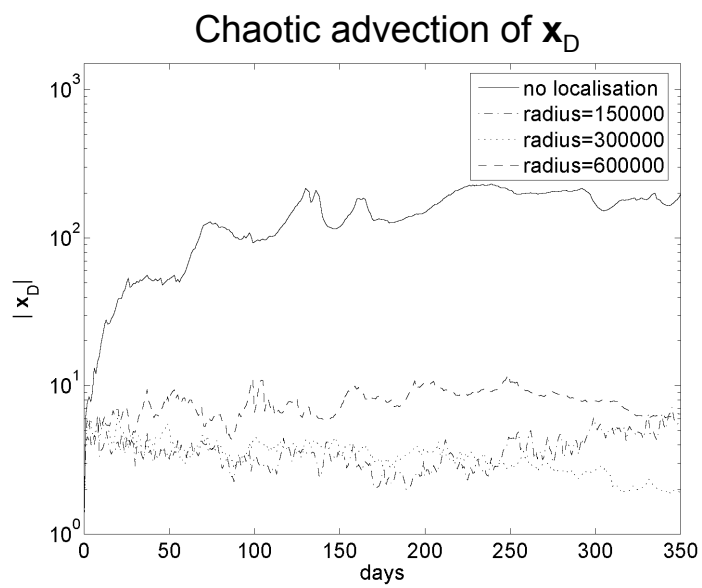
  - $\rho \circ ()$ is the function of distance between
    - Grid points
    - Grid and drifter
    - drifters

    denoting the Schur product (Hamill, 2001)
  - r_{loc} gives the radius of influence

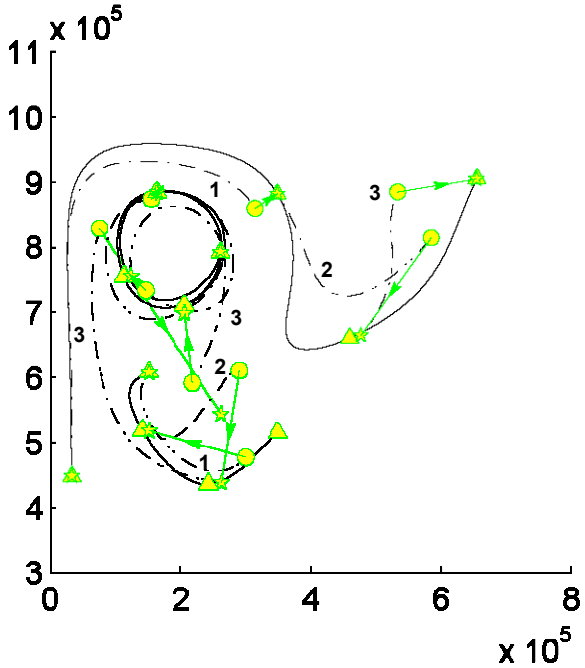
Effect of Localization r_{loc} for Turbulent Dynamics

- Three localization radii investigated
  - r_{loc}=150km, 300km, 600m
  - $(v, \Delta T, N_e)$ =(400m^{2}s^{-1}, 1days, 80) and 36 drifters
- Better convergence using the localization
  - Optimal r_{loc}=300km





Drifter Update Examples

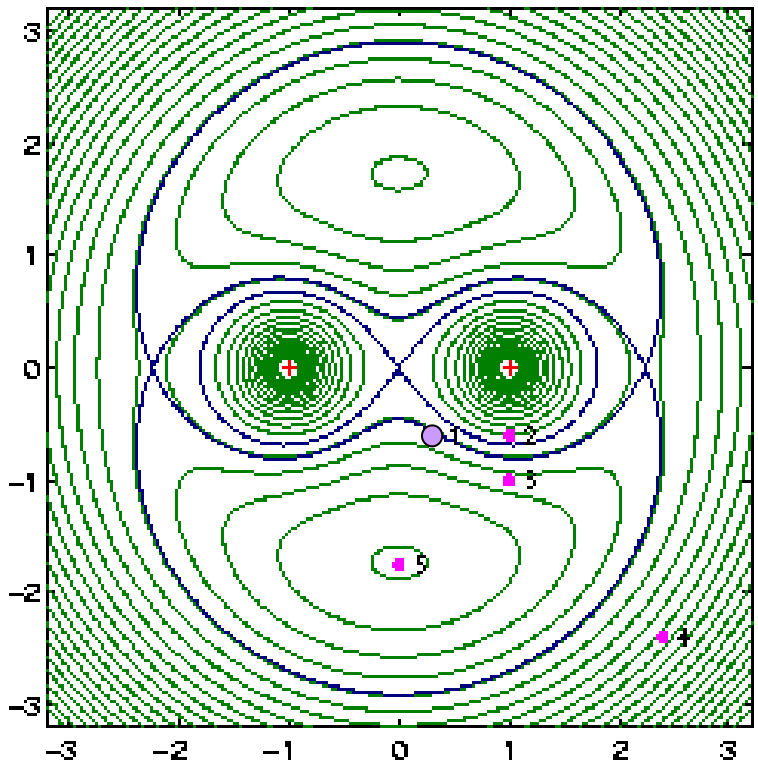


- Along the jet
  - Large derivation: can be still successful
- In the recirculating region
- In the eddies
  - Detrainment process by the saddle

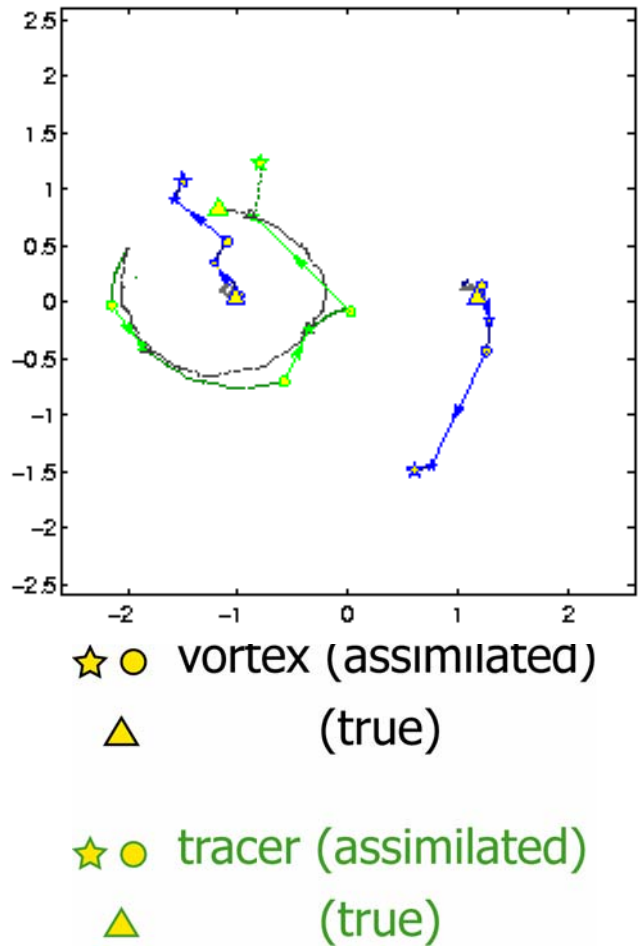
Can LaDA handle chaotic drifter dynamics?

Dealing with Lagrangian Saddle: Estimation of Coherent Structures

- Parameters (σ, ρ, ΔT)=(0.04, 0.02, 1.5)



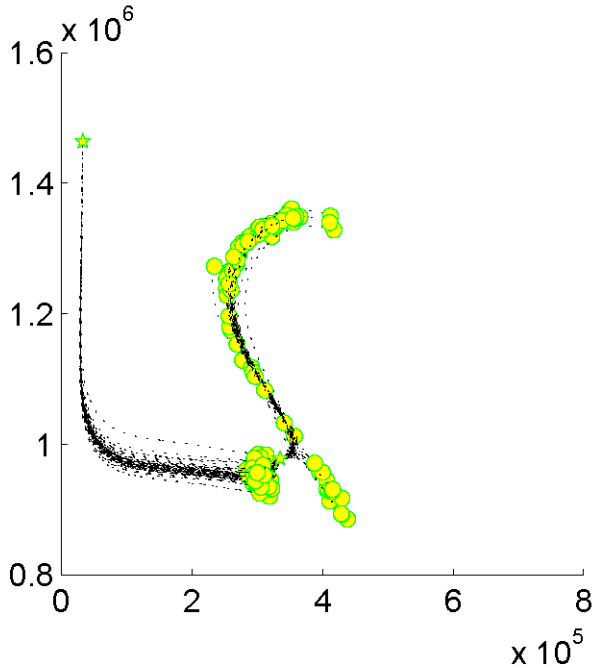
Ex: failuer case for I.C.no.3



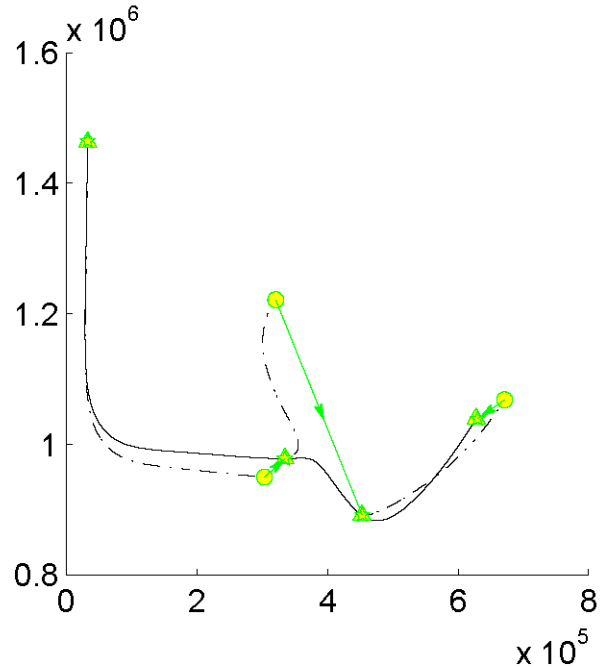
Saddle Effect with the Eulerian Model

- Large updates in drifter position occur near saddle point
  - Prior PDF distribution can be bimodal; Unimodal Gaussian distribution breaks down.
  - It can potentially produce large and spurious changes in the flow

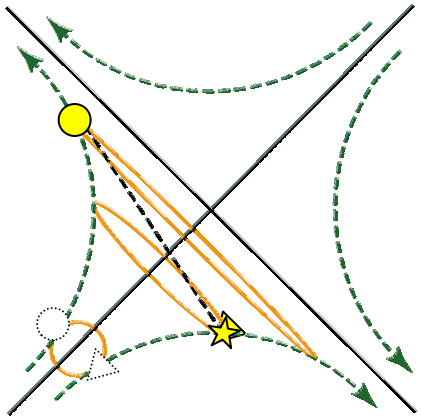
Ensemble Spread of drifter



Update mechanism of the mean



Saddle Effect



- Saddle effect occurs near the linearly hyperbolic region of velocity

  (λ: hyperbolicity given by the positive local Lyapunov exponent)

$$\frac{d}{dt}\mathbf{x}_D = \mathbf{M}_D(\mathbf{x}_F)\mathbf{x}_D \quad \approx \quad \mathbf{x}_D(t) = \mathbf{F}_D(\mathbf{x}_F)\mathbf{x}_D(t_0)$$

$$\frac{d}{dt}\mathbf{P}_{DD} = \mathbf{M}_D(\mathbf{x}_F)\mathbf{P}_{DD} + \mathbf{P}_{DD}\mathbf{M}_D(\mathbf{x}_F)^T \quad \approx \quad \mathbf{P}_{DD}(t) = \mathbf{F}_D(\mathbf{x}_F)\mathbf{P}_{DD}(t_0)\mathbf{F}_D(\mathbf{x}_F)^T$$

- $\Delta\mathbf{x}^a_D$ can be updated correctly: $\mathbf{P}_{DD}$ grows linearly in time with exponent 2λ:

$$\Delta\mathbf{x}^a_D = \mathbf{P}^f_{DD}\left(\mathbf{P}^f_{DD} + \mathbf{R}^o_D\right)^{-1}\left(\mathbf{y}^o_D - \mathbf{x}^f_D\right)$$

- $\Delta\mathbf{x}^a_F$ may be unreasonably large because

$$\Delta\mathbf{x}^a_F = \mathbf{P}^f_{FD}\left(\mathbf{P}^f_{DD} + \mathbf{R}^o_D\right)^{-1}\left(\mathbf{y}^o_D - \mathbf{x}^f_D\right)$$

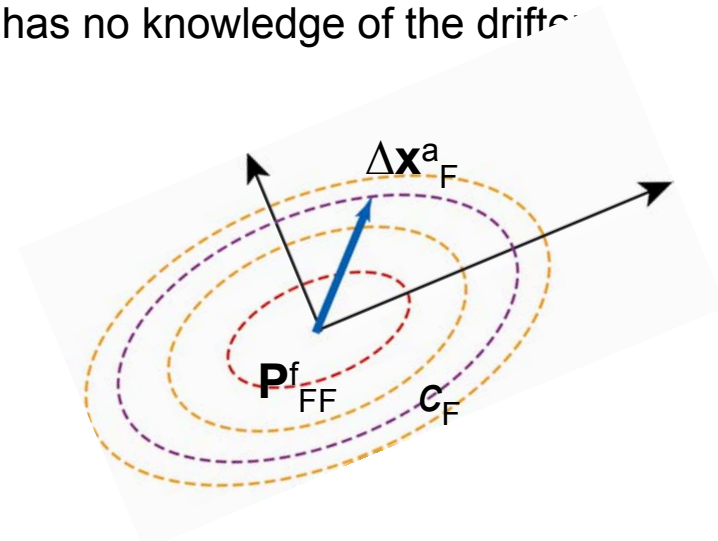
  - $\mathbf{P}_{FD}$ may be approximated by too large with exponent λ:
  - Dragged by a large innovation ($\mathbf{y}^o_D$-$\mathbf{x}^o_D$)

Tracer Control: Sanity check for $\Delta\mathbf{x}^a_F$

- $C_F \equiv$ Standard deviation of $\Delta\mathbf{x}^a_F$ with respect to the expected error
  - $\mathbf{P}^f_{FF}$ is independent of $\mathbf{x}^f_D$ or $\mathbf{x}^t_D$: the flow has no knowledge of the drifter

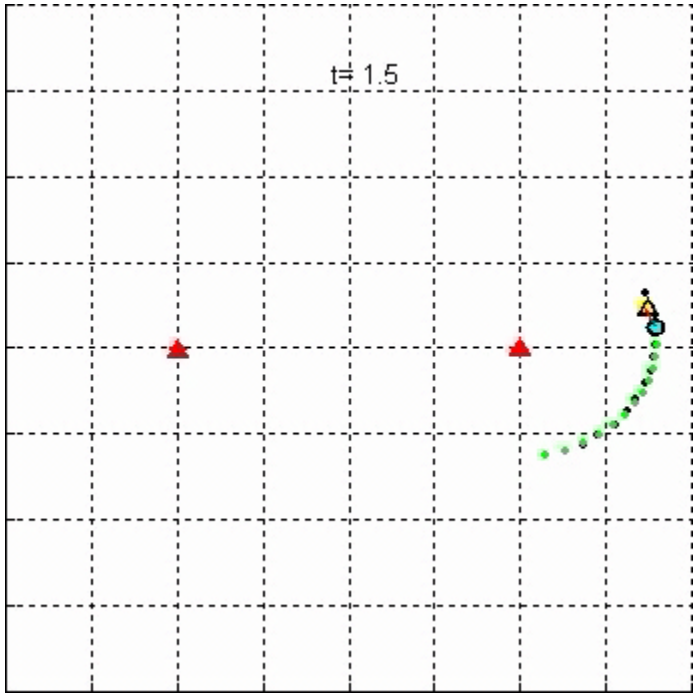
$$C_F = \left(\Delta\mathbf{x}^a_F\right)^T \left(\mathbf{P}^f_{FF}\right)^{-1} \Delta\mathbf{x}^a_F$$

$$\Delta\mathbf{x}^a_F = \mathbf{P}^f_{FD}\left(\mathbf{P}^f_{DD} + \mathbf{R}^o_D\right)^{-1}\left(\mathbf{y}^o_D - \mathbf{x}^f_D\right)$$



- Implementation
  - if $C_F < \delta$: Update of $\mathbf{x}_F$
  - if $C_F \geq \delta$: No update of $\mathbf{x}_F$ (but $\mathbf{x}_D$ is updated)
  - C_F is computed for $\mathbf{x}_D$ within r_{FD} from $\mathbf{x}_D$ ($r_{FD} < r_{loc}$)
  - Control is applied to entire $\mathbf{x}_D$ within r_{loc} from $\mathbf{x}_D$

Tracer Control with Vortex System



$\Delta T=1.5$ and $\delta=3$

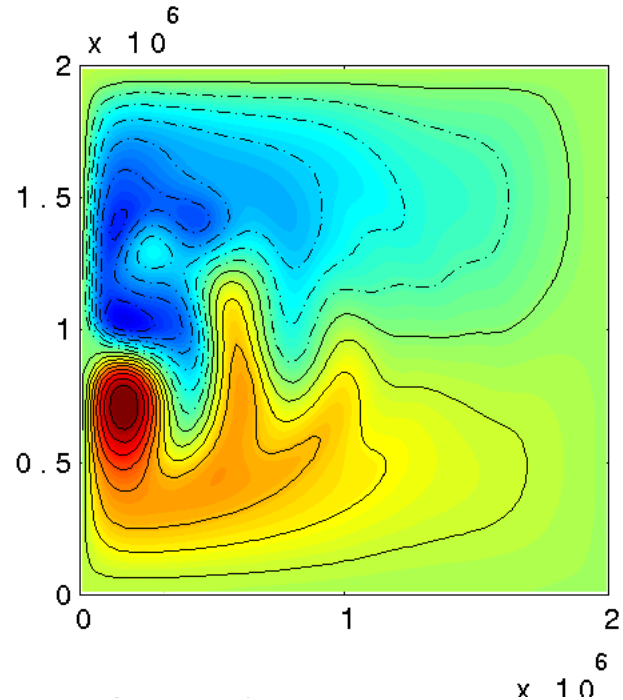
- EKF
  - Without TC
  - With TC
- EnKF
  - Without TC
  - With TC

Dynamical Systems Theory

- Two-dimensional drifter dynamics

$$\frac{d}{dt}x = u(x,y,t) = -\frac{\partial}{\partial y}\psi(x,y,t)\times a(x,y,t)$$
$$\frac{d}{dt}y = v(x,y,t) = \frac{\partial}{\partial x}\psi(x,y,t)\times a(x,y,t)$$

  - Velocity **u**(x,y,t), is tangent to the streamfunction Ψ(x,y,t)



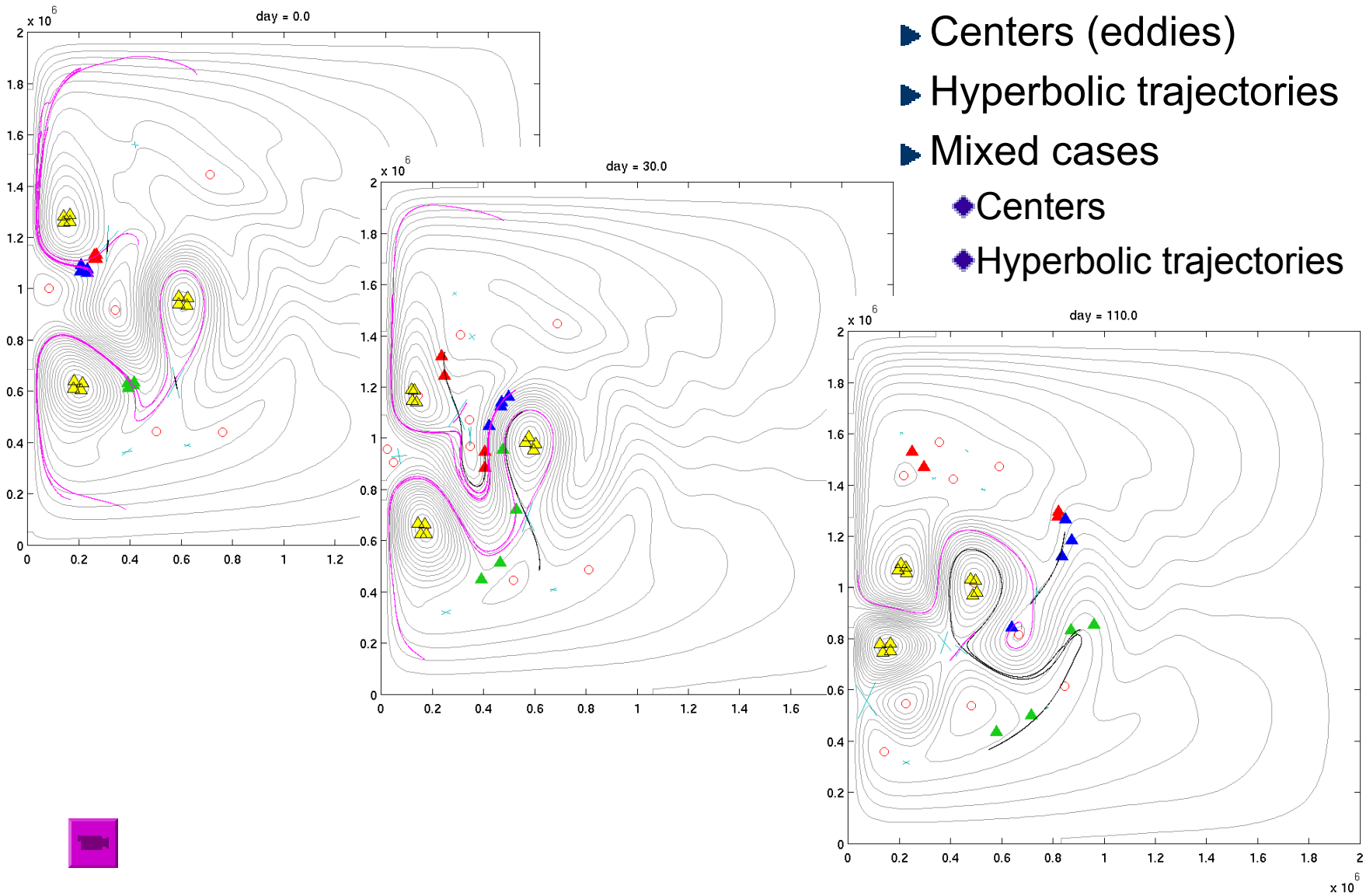
- Steady flow

$$\frac{d}{dt}\psi\left(x_D(t),y_D(t)\right)=0$$
$$\psi\left(x_D(t),y_D(t)\right)=\psi\left(x_D(t_0),y_D(t_0)\right)$$

  - Any trajectory remains on the iso- Ψ(x,y) curve
  - Streamfunction field Ψ(x,y) completely describes the glob
  - Stable and unstable trajectories from the hyperbolic fixed point (saddle) define the global template

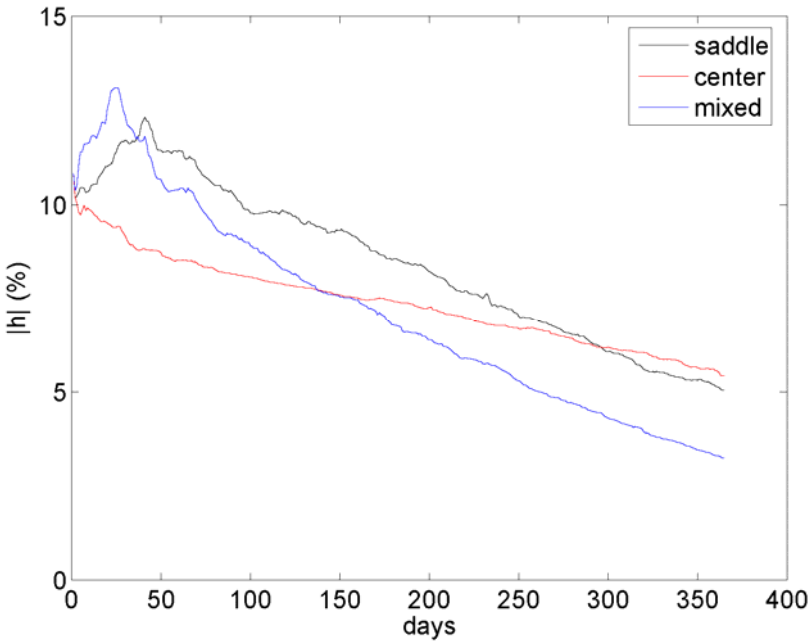
- Unsteady flow
  - Stable and unstable invariant manifolds (material curve) from the hyperbolic trajectory (Lagrangian saddle) define the global template of the Lagrangian dynamics

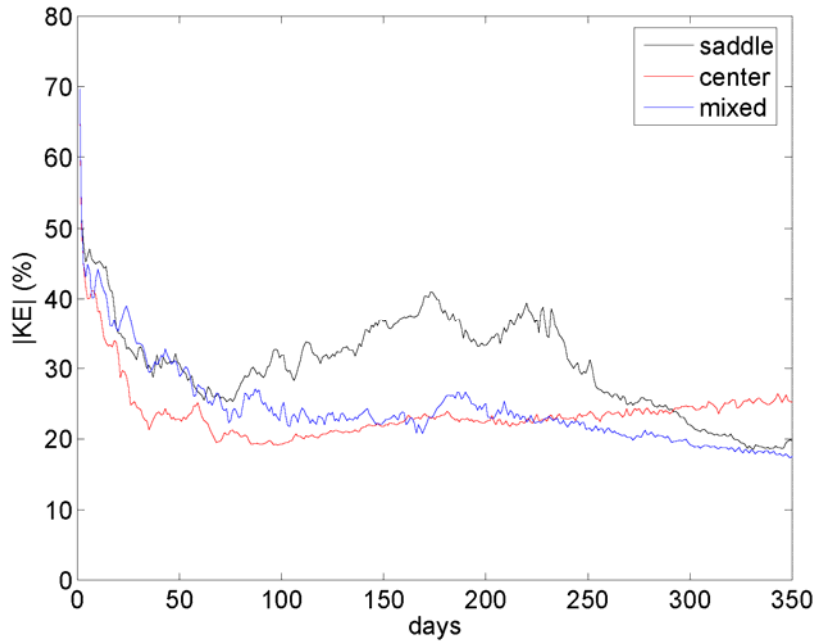
Targeted Observing System Design



- Centers (eddies)
- Hyperbolic trajectories
- Mixed cases
  - Centers
  - Hyperbolic trajectories

Preliminary Results





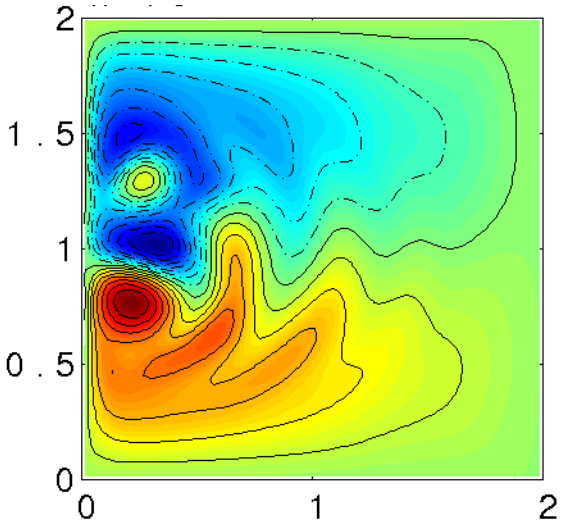
$$|h| = \sqrt{\sum_{i,j}^{Nx,Ny} \left(\bar{h}_{i,j} - h^t_{i,j}\right)^2 / \sum_{i,j}^{Nx,Ny} \left(h^t_{i,j}\right)^2}$$

$$|KE| = \sqrt{\sum_{i,j}^{Nx,Ny} \left(\bar{u}_{i,j} - u^t_{i,j}\right)^2 + \left(\bar{v}_{i,j} - v^t_{i,j}\right)^2 / \sum_{i,j}^{Nx,Ny} \left(u^t_{i,j}\right)^2 + \left(v^t_{i,j}\right)^2}$$

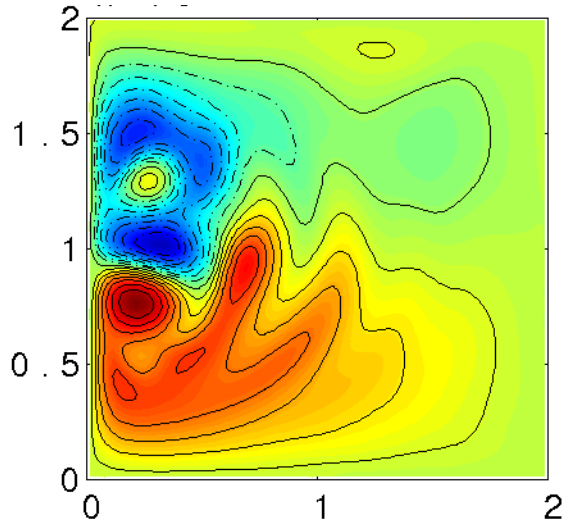
Targeted Observation System

T=365 days

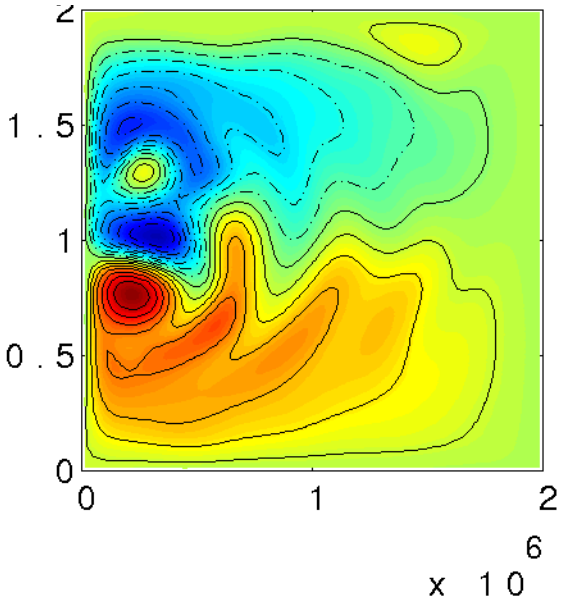
- Control



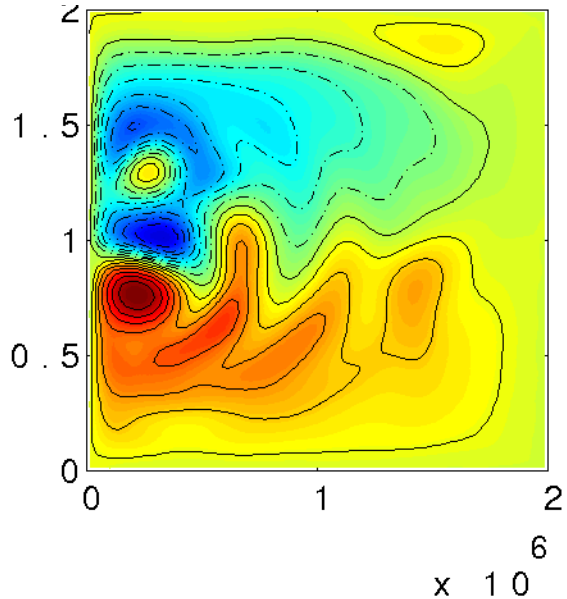
- Centers



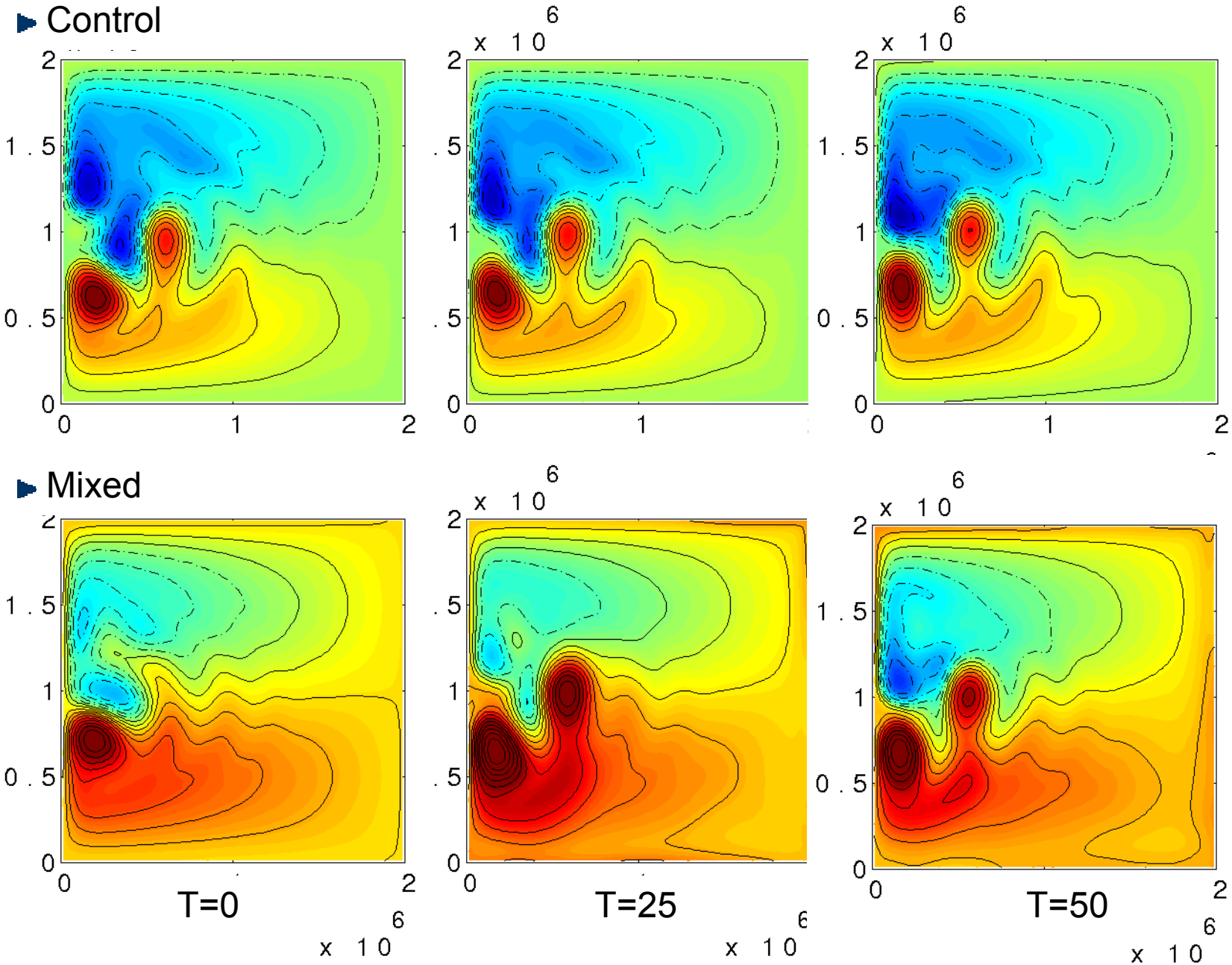
- Mixed



- Hyperbolic trajectories

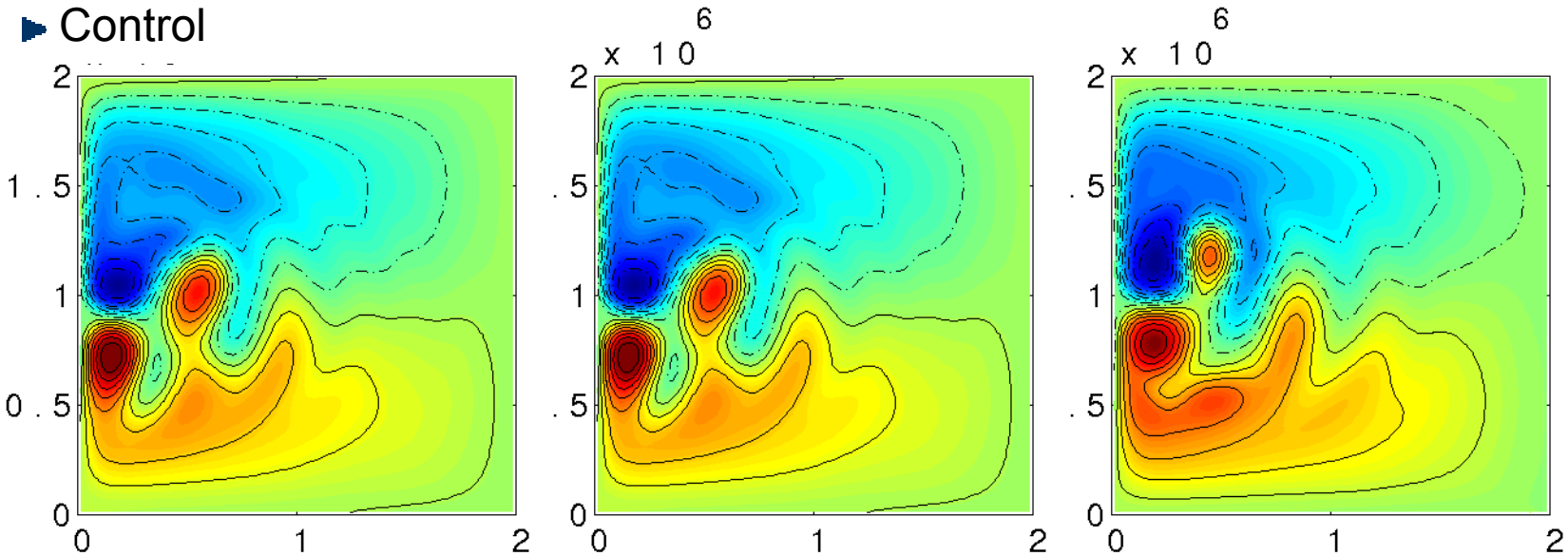


- Control

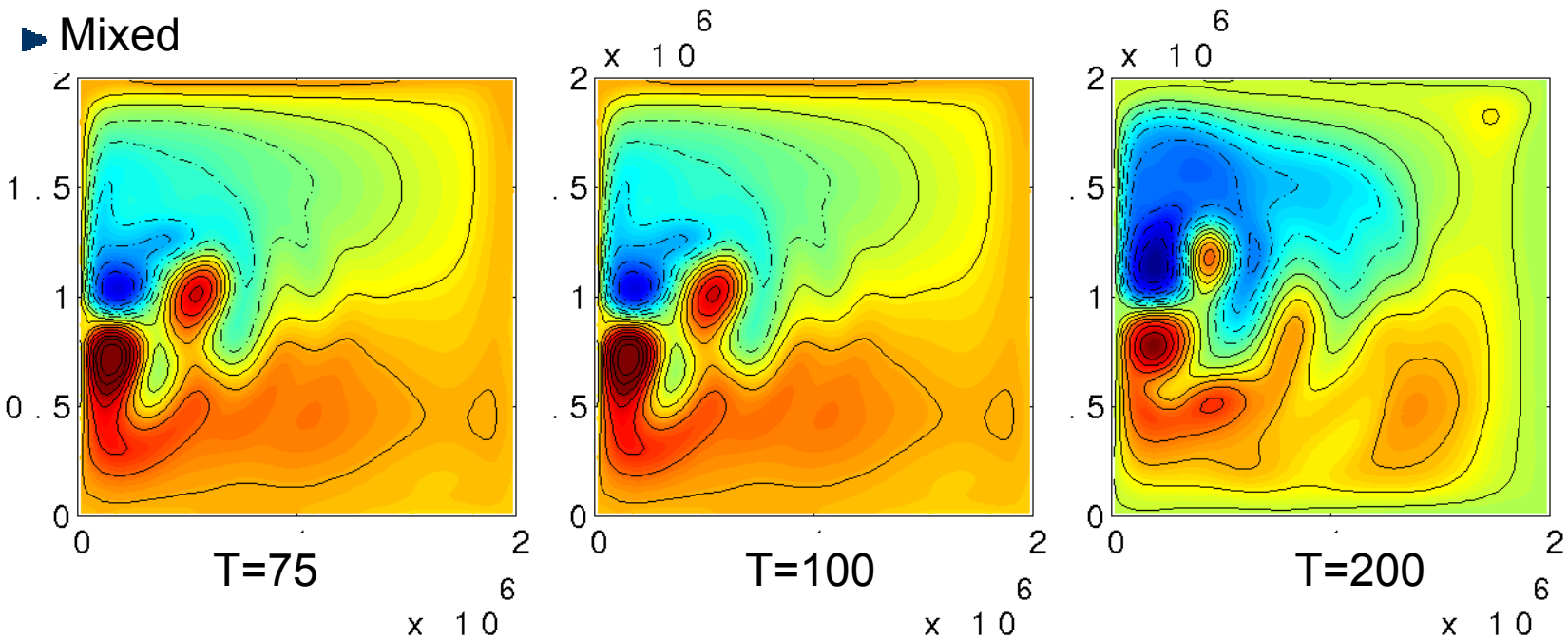


- Mixed

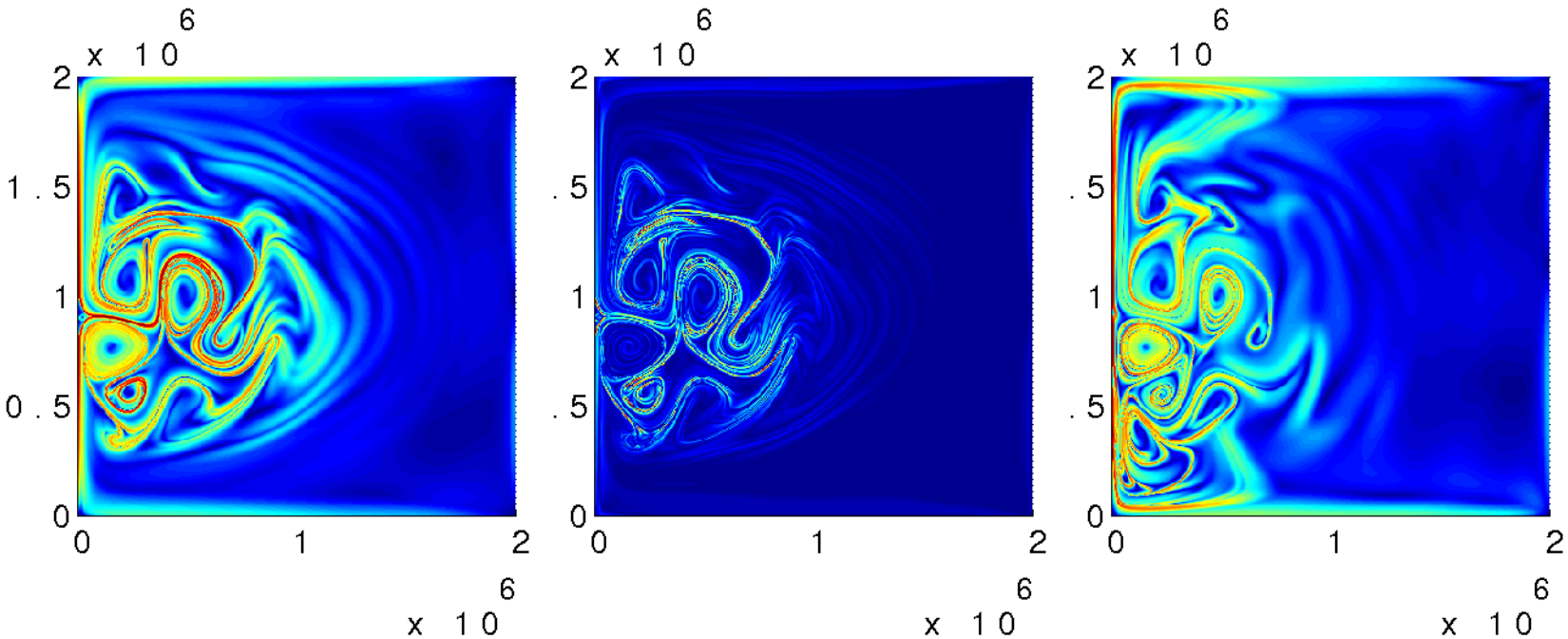
- Control



- Mixed



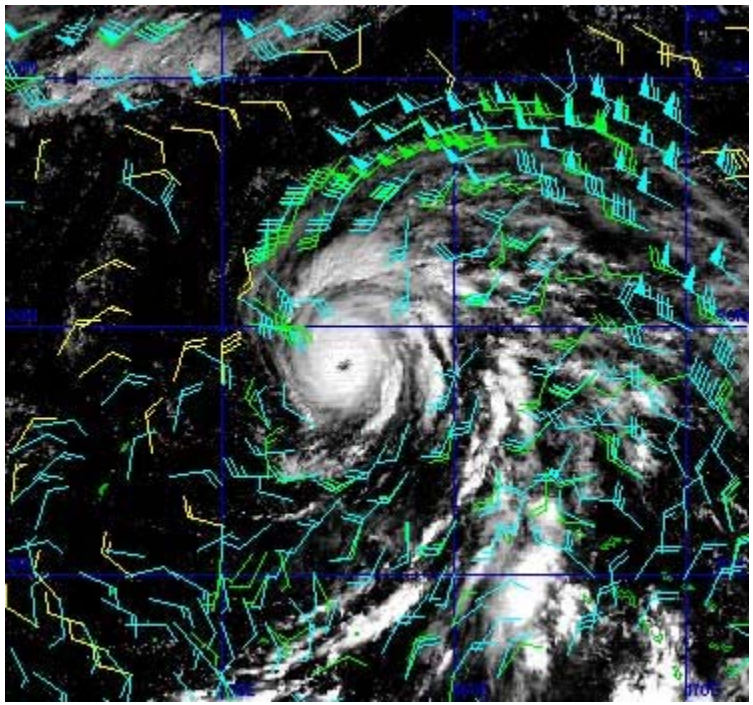
Observing System Design for Transient Flow Dynamics: Finite Time Lyapunov Exponents

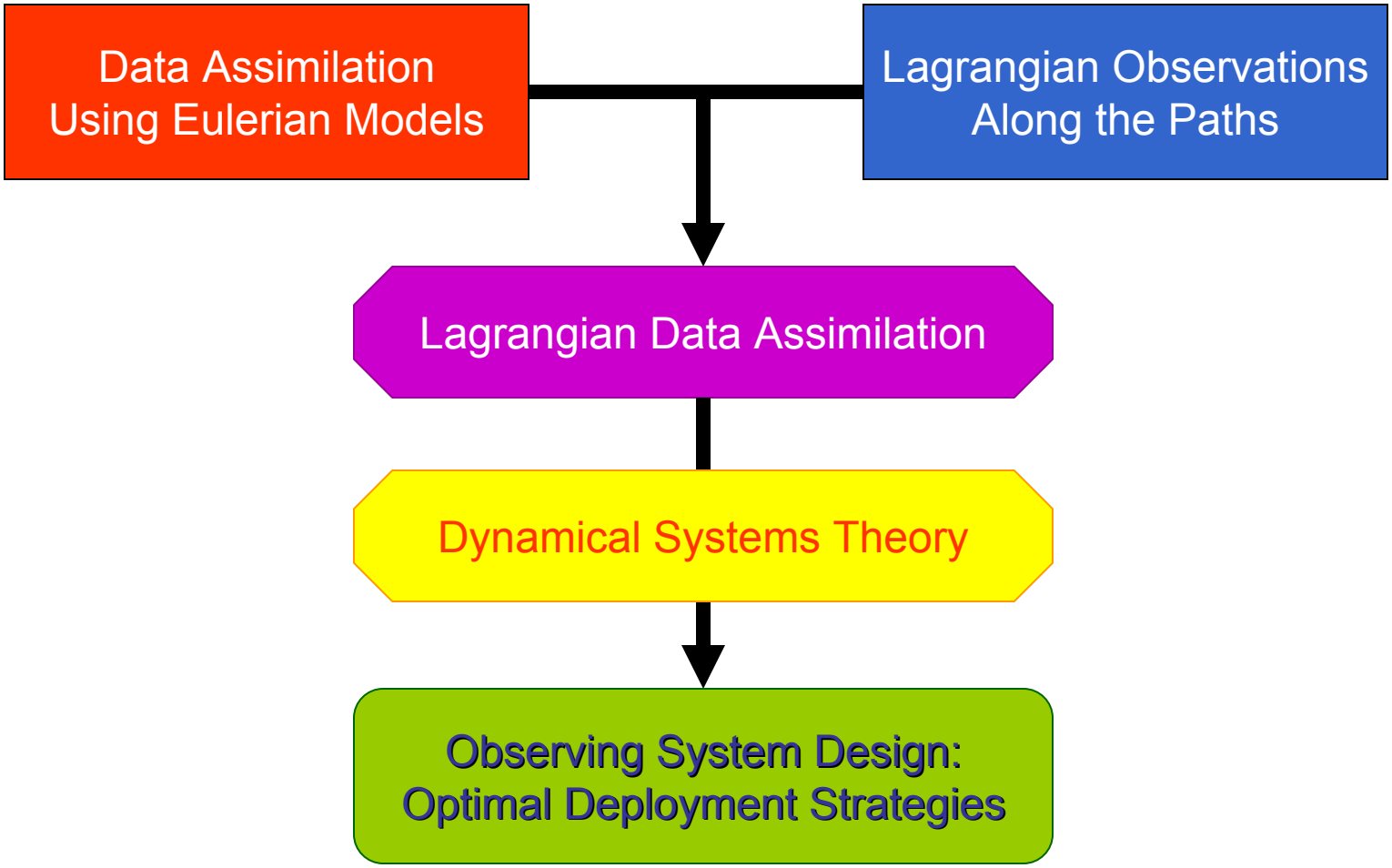


Future Directions: Atmospheric Applications
Atmospheric motion vector / Cloud wind vector

- Global wind vectors are used for the initial value of Numerical Weather Prediction (NWP), via data assimilation.

- Satellite-based observations provide wide coverage for the monitoring of atmospheric motion vectors by tracking the movement of clouds and interpolating the movement in time.



Idea: Use Lagrangian data assimilation method
& assimilate the cloud feature positions directly into the morel,
- without any interpolation in time
- cloud height information is naturally taken care of

Summary and Work in Progress