APPLICATION TO 3D MODELS
with a generalized vertical coordinate ζ

We can show

$$\frac{\partial z}{\partial \zeta}\underbrace{\left[\nabla_z \cdot (\rho v) + \frac{\partial}{\partial z}(\rho w)\right]}_{\text{3D mass divergence}} = \nabla_\zeta \cdot \left(\rho \mathbf{v} \frac{\partial z}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left[\rho\left(\frac{\partial z}{\partial t}\right)_\zeta + \rho\left(\dot{\zeta}\frac{\partial z}{\partial \zeta}\right)\right]$$

$\wr$

0 for non-acoustic motion

Continuity equation:

$$\left(\frac{\partial}{\partial t}\right)_\zeta\left(\rho\frac{\partial z}{\partial \zeta}\right) + \underbrace{\nabla_\zeta \cdot \left(\rho \mathbf{v} \frac{\partial z}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left[\rho\dot{\zeta}\left(\frac{\partial z}{\partial \zeta}\right)\right]} = 0$$

3D MFB Scheme: $\Downarrow$

$$\nabla_\zeta \cdot \left(\overline{\rho \mathbf{v} \frac{\partial z}{\partial \zeta}}\right) + \frac{\partial}{\partial \zeta}\left[\overline{\rho\left(\frac{\partial z}{\partial t}\right)_\zeta + \rho\left(\dot{\zeta}\frac{\partial z}{\partial \zeta}\right)}\right] - \frac{\partial}{\partial \zeta}\left[\rho\left(\frac{\partial z}{\partial t}\right)_\zeta\right]$$

Here the overbar denotes

- a horizontal weighted average for quasi-static models
  (iteration may be needed)
- a vertical weighted average for compressible cloud-resolving models.