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The Implementation of the Sigma Pressure Hybrid Coordinate into the GFS

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Hydrodynamics

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1 The Implementation of the Sigma Pressure Hybrid Coordinate into the GFS

The derivation of the hybrid coordinate is based on ECMWF's publications. The distance between the set of equations and the corresponding computer code is considerable and many details have to be specified and expanded. These overly detailed notes attempt to provide a DOCUMENTATION of what was actually derived and coded into the existing GFS over-structure. Attached to these notes is a calling tree of the various key routines of the hybrid GFS with accents on the new hybrid codes or the replacement of the old sigma codes.

Some of the equations contain partial vertical sums which may be delicate to code. In these cases a detailed exposition of the code is provided for a four level model so that all terms, especially near boundaries are accounted for. It is pointed out that the hybrid sigma-pressure coordinate presented here does not become a pure sigma structure as defined in the current GFS. Comparison of the tendencies produced by NCEP's sigma definition and the corresponding ECMWF definition reveals very similar forecasts at most model levels except at the very top of the model, as expected.

The reader is advised to refer to the notation and vertical structure pages at the end of this document before proceeding with the derivation.

Indexing in the defining equations is from the top of the atmosphere to bottom. Indexing in the GFS has always been from bottom to top and the reader is cautioned to exercise care when switching between notes and codes.

We begin with the discrete definition of a hybrid coordinate.

Let $\eta(0, p_s) = 0. \qquad \eta(p_s, p_s) = 1. \qquad \eta_{k+\frac{1}{2}} = \frac{A_{k+\frac{1}{2}}}{p_0} + B_{k+\frac{1}{2}}$

$A_{k+\frac{1}{2}}$ $B_{k+\frac{1}{2}}$ are scaled to model units, $k = 0$ at top of atmosphere, and $p_0 = 101.325$ cb. It is noted that the actual values of η are only required for semi-lagrange interpolations, not for the derivation of the model's equations.

Therefore, $A_{\frac{1}{2}} = B_{\frac{1}{2}} = 0.$ Let $(U, V) = \cos\theta\,(u, v)$

The model equations without diabatic forcing terms are:

$$\frac{\partial U}{\partial t} + \frac{1}{a\cos^2\theta}\left(U\frac{\partial U}{\partial \lambda} + V\cos\theta\frac{\partial U}{\partial \theta}\right) + \dot{\eta}\frac{\partial U}{\partial \eta}$$

$$-fV + \frac{1}{a}\left(\frac{\partial \phi}{\partial \lambda} + RT_v\frac{\partial}{\partial \lambda}\ln p\right) = 0 \tag{1.0.1}$$

$$\frac{\partial V}{\partial t} + \frac{1}{a\cos^2\theta}\left(U\frac{\partial V}{\partial \lambda} + V\cos\theta\frac{\partial V}{\partial \theta} + \sin\theta\left(U^2 + V^2\right)\right) + \dot{\eta}\frac{\partial V}{\partial \eta}$$

$$+fU + \frac{\cos\theta}{a}\left(\frac{\partial \phi}{\partial \theta} + RT_v\frac{\partial}{\partial \theta}\ln p\right) = 0 \tag{1.0.2}$$

$$\frac{\partial T_v}{\partial t} + \frac{1}{a\cos^2\theta}\left(U\frac{\partial T_v}{\partial \lambda} + V\cos\theta\frac{\partial T_v}{\partial \theta}\right) + \dot{\eta}\frac{\partial T_v}{\partial \eta}$$

$$= \frac{\kappa T_v \omega}{p}\left[\frac{1 + \epsilon q}{1 + (\delta - 1)\,q}\right] \tag{1.0.3}$$

For eq. (1.0.3) see treatment of the thermodynamic equation.

The moisture specific humidity equation is:

$$\frac{\partial q}{\partial t} + \frac{1}{a\cos^2\theta}\left(U\frac{\partial q}{\partial \lambda} + V\cos\theta\frac{\partial q}{\partial \theta}\right) + \dot{\eta}\frac{\partial q}{\partial \eta} = 0 \tag{1.0.4}$$

and the continuity equation is written as:

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right)+\nabla\cdot v_H\frac{\partial p}{\partial \eta}+\frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)=0 \tag{1.0.5}$$

$v_H=(u,v)$ unscaled

The hydrostatic equation is:

$$\frac{\partial \phi}{\partial \eta}=-\frac{R_dT_v}{p}\frac{\partial p}{\partial \eta} \tag{1.0.6}$$

and ω is diagnosed from:

$$\omega=-\int_0^\eta \nabla\cdot\left(v_H\frac{\partial p}{\partial \eta}\right)\mathrm{d}\eta+v_H\cdot\nabla p \tag{1.0.7}$$

eq. (1.0.7) follows from $\quad \omega=\frac{\mathrm{d}p}{\mathrm{d}t}=\frac{\partial p}{\partial t}+v_H\cdot\nabla p+\dot{\eta}\frac{\partial p}{\partial \eta}$

substitute $\quad \frac{\partial p}{\partial t}\quad$ in eq. (1.0.5), then

$\frac{\partial}{\partial \eta}\left(\omega-v_H\cdot\nabla p-\dot{\eta}\frac{\partial p}{\partial \eta}\right)+\nabla\cdot v_H\frac{\partial p}{\partial \eta}+\frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)=0$

integrate $\quad 0$ to $\eta \quad (\omega-v_H\cdot\nabla p)\mid_0^\eta+\int_0^\eta\nabla\cdot v_H\frac{\partial p}{\partial \eta}\mathrm{d}\eta=0$

if $\quad \omega_{\eta=0}=0 \quad$ and $\quad (v_H\cdot\nabla p)_{\eta=0}=0 \quad$ eq. (1.0.7) follows.

An equation for $\quad \frac{\partial p_s}{\partial t} \quad$:

integrate eq. (1.0.5) $\int_0^1 (\ )\,\mathrm{d}\eta$ and use $\dot{\eta}(0) = \dot{\eta}(1) = 0$

$\int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right)\mathrm{d}\eta + \int_0^1 \left(\nabla \cdot v_H \frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)\right)\mathrm{d}\eta = 0$

$\frac{\partial p}{\partial t}\Big|_0^1 + \int_0^1 \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta = 0$. If $\left(\frac{\partial p}{\partial t}\right)_{\eta=0} = 0$ and $\left(\frac{\partial p}{\partial t}\right)_{\eta=1} = \frac{\partial p_s}{\partial t}$

$$\frac{\partial p_s}{\partial t} = -\int_0^1 \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta \tag{1.0.8}$$

or:

$$\frac{\partial \ln p_s}{\partial t} = -\frac{1}{p_s}\int_0^1 \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta$$

For the vertical advection terms we need an expression for $\dot{\eta}\frac{\partial p}{\partial \eta}$:
integrate eq. (1.0.5) $\int_0^\eta (\ )\,\mathrm{d}\eta$, then

$\int_0^\eta \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right)\mathrm{d}\eta + \int_0^\eta \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta + \int_0^\eta \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)\mathrm{d}\eta = 0$

$\frac{\partial}{\partial t}(p - p_0) + \int_0^\eta \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta + \dot{\eta}\frac{\partial p}{\partial \eta} = 0$ since $\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{\eta=0} = 0$

$$\dot{\eta}\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial t} - \int_0^\eta \nabla \cdot v_H \frac{\partial p}{\partial \eta}\mathrm{d}\eta \tag{1.0.9}$$

1.1 Vertical Discretization

Define at interfaces

$$p_{k+\frac{1}{2}} = A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} p_s \tag{1.1.1}$$

The surface pressure eq. (1.0.8) becomes:

$\frac{\partial \ln p_s}{\partial t} = -\frac{1}{p_s} \sum_{k=1}^{levs} \left(\nabla \cdot v_{Hk} \Delta p_k \right) \ , \qquad \Delta p_k = p_{k+\frac{1}{2}} - p_{k-\frac{1}{2}} > 0$

note that v_{Hk} is the unscaled horizontal velocity at layer k

let $D_k = \nabla \cdot v_{Hk}$, then,

$$\frac{\partial \ln p_s}{\partial t} = -\frac{1}{p_s} \sum_{k=1}^{levs} \nabla \cdot v_{Hk} \Delta p_k \qquad (1.1.2)$$

$$= -\frac{1}{p_s} \sum \left[D_k \Delta p_k + v_{Hk} \cdot \nabla \left(\Delta p_k \right) \right]$$

$$= -\sum \left[\frac{D_k}{p_s} \Delta p_k + v_{Hk} \cdot \left(\frac{1}{p_s} \nabla \Delta p_k \right) \right]$$

We want to express pressure dependent quantities in terms of the natural log of surface pressure, therefore:

$\frac{1}{p_s} \nabla \Delta p_k = \frac{1}{p_s} \nabla \left[\left(A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} p_s \right) - \left(A_{k-\frac{1}{2}} + B_{k-\frac{1}{2}} p_s \right) \right]$

$= \Delta B_k \nabla \ln p_s \quad \text{where} \quad \Delta B_k = B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}}$

$\frac{\partial \ln p_s}{\partial t} = -\sum_{k=1}^{levs} \left(\frac{D_k}{p_s} \Delta p_k + \Delta B_k v_{Hk} \cdot \nabla \ln p_s \right)$

Similarly from eq. (1.0.9) at interface k

$\left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k+\frac{1}{2}} = -\frac{\partial}{\partial t} p_{k+\frac{1}{2}} - \sum_{j=1}^{k} \nabla \cdot v_{Hj} \Delta p_j$

$= -B_{k+\frac{1}{2}} \frac{\partial p_s}{\partial t} - \sum_{j=1}^{k} \nabla \cdot v_{Hj} \Delta p_j$

$= -p_s \left(B_{k+\frac{1}{2}} \frac{1}{p_s} \frac{\partial p_s}{\partial t} + \frac{1}{p_s} \sum_{j=1}^{k} \nabla \cdot v_{Hj} \Delta p_j \right)$

$= -p_s \left[B_{k+\frac{1}{2}} \frac{\partial \ln p_s}{\partial t} + \frac{1}{p_s} \sum_{j=1}^{k} \left(D_j \Delta p_j + v_{Hj} \cdot \nabla \Delta p_j \right) \right]$

$$= -p_s \left[B_{k+\frac{1}{2}} \frac{\partial \ln p_s}{\partial t} + \sum_{j=1}^{k} \left(\frac{D_j}{p_s} \Delta p_j + \Delta B_j v_{Hj} \cdot \nabla \ln p_s \right) \right] \tag{1.1.3}$$

1.2 Vertical Advection

For conservation of angular momentum, the vertical advection of scalar x at layer k is:

$$\left(\dot{\eta} \frac{\partial x}{\partial \eta} \right)_k = \frac{1}{2 \Delta p_k} \left[\left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k+\frac{1}{2}} (x_{k+1} - x_k) + \left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k-\frac{1}{2}} (x_k - x_{k-1}) \right] \tag{1.2.1}$$

For the hydrostatic eq. we have:

$\phi_{k+\frac{1}{2}} - \phi_{k-\frac{1}{2}} = -R_d \left(T_v \right)_k \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right)$

or:

$\phi_{k+\frac{1}{2}} = \phi_s + \sum_{j=k+1}^{levs} R_d \left(T_v \right)_j \ln \left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right)$

Since we need ϕ_k in layers, we define α_k as in Simmons and Burrage,

$$\phi_k = \phi_{k+\frac{1}{2}} + \alpha_k R \left(T_v\right)_k \tag{1.2.2}$$

$$\alpha_k = 1 - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \tag{1.2.3}$$

for $k \geq 1$, and $\alpha_1 = \ln 2$. (another choice could be $\alpha = 1$)

The implication of the choice of α_k affects the computation of $\nabla\phi$. This choice will not in general result in the sigma system used at NCEP (when $A_k \equiv 0$). The semi-implicit matrices will also be different and will be rederived.

1.3 The Pressure Gradient Term

$R_d T_v \nabla \ln p$ is discretized as follows:

$$R_d \left(T_v \nabla \ln p\right)_k = \frac{R_d (T_v)_k}{\Delta p_k} \left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \nabla p_{k-\frac{1}{2}} + \alpha_{k\nabla\Delta p_k} \right] \tag{1.3.1}$$

The above choice of α_k in eq. (1.2.3) will reduce eq. (1.3.1) to $RT_k \nabla \ln p_s$ when $A_k = 0$. In this case:

$$\tfrac{RT}{p} \nabla p_k = R_d T_k \left\{ \tfrac{1}{\Delta p_k} \left(p_{k+\frac{1}{2}} \ln p_{k+\frac{1}{2}} - p_{k-\frac{1}{2}} \ln p_{k-\frac{1}{2}} \right) \right\}$$

which is a discrete analog of $R_d T \nabla \frac{\partial}{\partial p} \left(p \ln p\right) = R_d T \nabla \ln p$

1.4 The Time Discretization of the Momentum Equations

let: $\delta_t = \frac{x^+ - x^-}{2\Delta t}$ $\Delta_{tt} = x^+ - 2x + x^-$, T_r is a reference Temp.

let: $A\left(x\right) = \frac{1}{a\cos^2\theta}\left(U\frac{\partial x}{\partial\lambda} + V\cos\theta\frac{\partial x}{\partial\theta}\right) + \dot{\eta}\frac{\partial x}{\partial\eta}$ (3D advection)

then from eq. (1.0.1) and eq. (1.0.2)

$$\delta_t U = -A\left(U\right) + fV - \frac{1}{a}\left(\frac{\partial\phi}{\partial\lambda} + RT_v\frac{\partial\ln p}{\partial\lambda}\right) - \frac{\beta}{2a}\Delta_{tt}\left(Y\frac{\partial T}{\partial\lambda} + RT_r\frac{\partial}{\partial\lambda}\ln p_s\right)$$

$$\delta_t V = -A\left(V\right) - fU - \frac{\cos\theta}{a}\left(\frac{\partial\phi}{\partial\theta} + RT_v\frac{\partial\ln p}{\partial\theta}\right) - \frac{\sin\theta}{a\cos^2\theta}\left(U^2 + V^2\right)$$

$$-\frac{\beta\cos\theta}{2a}\Delta_{tt}\left(Y\frac{\partial T}{\partial\theta} + RT_r\frac{\partial}{\partial\theta}\ln p_s\right)$$

The semi-implicit terms are treated as follows:

$$\frac{\partial x}{\partial t} = \left(\frac{\partial x}{\partial t}\right)_{\text{explicit}} + \frac{\beta}{2}\left(L^+ + L^- - 2L\right)$$

$$= \left(\frac{\partial x}{\partial t}\right)_{\text{explicit}} + \frac{\beta}{2}\Delta_{tt}L$$

where L is the linear part of the tendency.

Now let $\left(A\left(U\right)\right)_k = \text{ADU}$ $\left(A\left(V\right)\right)_k = \text{ADV}$

$$\text{ADU} = \frac{1}{a\cos^2\theta}\left(U_k\frac{\partial U_k}{\partial\lambda} + V_k\cos\theta\frac{\partial U_k}{\partial\theta}\right) + \frac{1}{2\Delta p_k}\left[\left(\dot{\eta}\frac{\partial p}{\partial\eta}\right)_{k+\frac{1}{2}}\left(U_{k+1} - U_k\right) + \left(\dot{\eta}\frac{\partial p}{\partial\eta}\right)_{k-\frac{1}{2}}\left(U_k - U_{k-1}\right)\right]$$

$$\text{ADV} = \frac{1}{a\cos^2\theta}\left(U_k\frac{\partial V_k}{\partial\lambda} + V_k\cos\theta\frac{\partial V_k}{\partial\theta}\right) + \frac{1}{2\Delta p_k}\left[\left(\dot{\eta}\frac{\partial p}{\partial\eta}\right)_{k+\frac{1}{2}}\left(V_{k+1} - V_k\right) + \left(\dot{\eta}\frac{\partial p}{\partial\eta}\right)_{k-\frac{1}{2}}\left(V_k - V_{k-1}\right)\right]$$

Also define coding variables as follows:

$$-\cos\theta\nabla\phi_k = \begin{cases} -\frac{1}{a}\left(\frac{\partial\phi}{\partial\lambda}\right)_k & = \quad \text{UPHI} \\ \\ -\frac{\cos\theta}{a}\left(\frac{\partial\phi}{\partial\theta}\right)_k & = \quad \text{VPHI} \end{cases}$$

$$-RT_v\cos\theta\nabla\ln p = \begin{cases} -\frac{R}{a}\left(T_v\frac{\partial}{\partial\lambda}\ln p\right) & = \quad \text{UPRS} \\ \\ -\frac{R\cos\theta}{a}T_v\frac{\partial\ln p}{\partial\theta} & = \quad \text{VPRS} \end{cases}$$

then for coding purposes:

$$\delta_t U = \text{dudt} = -\text{ADU} \ + fV + \text{UPHI} + \text{UPRS} - \tfrac{\beta}{2a}\cdot\Delta_{tt}\left(Y\tfrac{\partial T}{\partial\lambda} + RT_r\tfrac{\partial\ln p_s}{\partial\lambda}\right)$$

$$\delta_t V = \text{dvdt} = -\text{ADV} - fU + \text{VPHI} + \text{VPRS} - \tfrac{\beta\cos\theta}{2a}\Delta_{tt}\left(Y\tfrac{\partial T}{\partial\theta} + RT_r\tfrac{\partial\ln p_s}{\partial\theta}\right)$$

$$-\tfrac{\sin\theta}{a\cos^2\theta}\left(U^2+V^2\right) \quad .$$

1.5 The Thermodynamic Equation

Conservation of energy and hydrostatic conditions imply:

$$\mathrm{d}Q = c_p\mathrm{d}T - \alpha\mathrm{d}p \ , \quad \alpha = \tfrac{1}{\rho} = \tfrac{RT_v}{p}$$

If $\mathrm{d}Q = 0$

$$c_p\tfrac{\mathrm{d}T}{\mathrm{d}t} = \alpha\tfrac{\mathrm{d}p}{\mathrm{d}t}$$

$$\tfrac{\mathrm{d}T}{\mathrm{d}t} = \tfrac{\alpha}{c_p}\tfrac{\mathrm{d}p}{\mathrm{d}t} = \tfrac{\alpha\omega}{c_p} = \tfrac{RT_v\omega}{c_p p} \ , \quad \alpha = \tfrac{RT_v}{p}$$

We also have

$$c_p = C_{pd} + (C_{pv} - C_{pd})\, q = C_{pd}\left[1 + \left(\frac{C_{pv}}{C_{pd}} - 1\right) q\right]$$

and $\kappa = \frac{R}{C_{pd}}$. We may then write eq. (1.0.3) as:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{RT_v\omega}{C_{pd}\left[1+\left(\frac{C_{pv}}{C_{pd}}-1\right)q\right]p} = \frac{\kappa T_v\omega}{(1+(\delta-1)q)p}$$

where $\delta = \frac{C_{pv}}{C_{pd}} = \frac{1846}{1004.6} = 1.837$

$$\delta - 1 = 0.8375$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\kappa T_v\omega}{[1+(\delta-1)q]p}$$

For T_v as history variable: let $\epsilon = \frac{R_v}{R_d} - 1 = 0.6077$

$$T_v = (1+\epsilon q)\,T\ , \quad \frac{\mathrm{d}T_v}{\mathrm{d}t} = (1+\epsilon q)\,\frac{\mathrm{d}T}{\mathrm{d}t} + \epsilon T\frac{\mathrm{d}q}{\mathrm{d}t}$$

$$\frac{\mathrm{d}T_v}{\mathrm{d}t} = (1+\epsilon q)\,\frac{\kappa T_v\omega}{(1+(\delta-1)q)p} + \epsilon T\frac{\mathrm{d}q}{\mathrm{d}t}$$

$$\frac{\mathrm{d}T_v}{\mathrm{d}t} = \underset{\substack{\Uparrow \\ \text{calculated in dynamics}}}{\frac{\kappa T_v\omega}{p}\frac{(1+\epsilon q)}{(1+(\delta-1)q)}} + \underset{\substack{\Uparrow \\ \text{calculated in physics}}}{\epsilon T\frac{\mathrm{d}q}{\mathrm{d}t}}$$

The “correction” term $\frac{1+\epsilon q}{1+(\delta-1)q} = \frac{1+0.6077q}{1+0.8375q}$

for large $q = \frac{50}{1000}$

$\frac{1+\epsilon q}{1+(\delta-1)q} = 0.99$

1.6 The Thermodynamic Energy Conversion Term

We will use the ECMWF energy conversion term (Ritchie 95, eq. 2.25).

Let $C_q = \frac{1+\epsilon q}{1+(\delta-1)q}$, the ECMWF energy conv. term is:

$$\left(\kappa \tfrac{T_v \omega}{p} \tfrac{1+\epsilon q}{1+(\delta-1)q}\right)_k = \kappa \left(T_v\right)_k \tfrac{1+\epsilon q_k}{1+(\delta-1)q_k} \Bigg\langle$$

$$-\tfrac{1}{\Delta p_k} \left\{ \ln\left(\tfrac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \textstyle\sum_{j=1}^{k-1} \left[D_j \Delta p_j + p_s \left(v_j \cdot \nabla \ln p_s\right) \Delta B_j\right] \right.$$

$$\left. + \ \alpha_k \left[D_k \Delta p_k + p_s \left(v_k \cdot \nabla \ln p_s\right) \Delta B_k\right] \right\}$$

$$+\frac{p_s}{\Delta p_k} \left[\Delta B_k + \frac{C_k}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\right] \left(v_k \cdot \nabla \ln p_s\right) \Bigg\rangle \qquad (1.6.1)$$

where

$$C_k = A_{k+\frac{1}{2}} B_{k-\frac{1}{2}} - A_{k-\frac{1}{2}} B_{k+\frac{1}{2}}$$

The dynamics grid computations are performed in subroutine gfidi. We now define variables for coding purposes. These variables are components of eq. (1.6.1).

$$\text{worka}_k = \tfrac{\kappa (T_v)_k}{\Delta p_k} \cdot \tfrac{1+\epsilon q_k}{1+(\delta-1)q_k}$$

$$\text{workb}_k = \ln\left(\tfrac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \textstyle\sum_{j=1}^{k-1} \left[D_j \Delta p_j + p_s \left(v_j \cdot \nabla \ln p_s\right) \Delta B_j\right]$$

$$+ \alpha_k \left[D_k \Delta p_k + p_s \left(v_k \cdot \nabla \ln p_s\right) \Delta B_k\right]$$

$$\text{workc}_k = p_s \left[\Delta B_k + \tfrac{C_k}{\Delta p_k} \ln\left(\tfrac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\right] v_k \cdot \nabla \ln p_s$$

Noting that $\frac{1}{\Delta p_k}$ was absorbed in worka_k we have:

$$\left(\kappa T_v \omega \frac{1+\epsilon q}{1+(\delta-1)q}\right)_k = \mathrm{worka}_k \left(-\mathrm{workb}_k + \mathrm{workc}_k\right)$$

and the full thermodynamic equation (Ritchie 95, eq. 2.30) can now be programmed:

$$\frac{T_v^+ - T_v^-}{2\Delta t} = \left(\frac{\kappa T_v \omega (1+\epsilon q)}{1+(\delta-1)q}\right)_k - \frac{1}{a\cos^2\theta}\left[U_k \frac{\partial T_{v_k}}{\partial \lambda} + V_k \cos\theta \frac{\partial T_{v_k}}{\partial \theta}\right]$$

$$-\frac{1}{2\Delta p_k}\left[\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} \left(T_{v_{k+1}} - T_{v_k}\right) + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}} \left(T_{v_k} - T_{v_{k-1}}\right)\right] - \frac{\beta}{2}\Delta_{tt}\left(\tau D\right)_k$$

The matrix τ is defined by:

$$\left(\tau D\right)_k = \kappa T_r \left(\frac{1}{\Delta p_k^r} \ln\left(\frac{p^r_{k+\frac{1}{2}}}{p^r_{k-\frac{1}{2}}}\right) \sum_{j=1}^{k-1} D_j \Delta p_j^r + \alpha_k^r D_k\right)$$

and

$$\tilde{D} = \left(D_1 \ \ldots \ D_{Levs}\right)$$

1.7 Calculation of (UPRS, VPRS)

In order to maintain conservation properties in the discretized formulation, the expression of the pressure force term requires the following form:

$$R_d \left(T_v \cos\theta \nabla \ln p\right)_k = \frac{R_d \cos\theta (T_v)_k}{\Delta p_k}\left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \nabla p_{k-\frac{1}{2}} + \alpha_k \nabla \Delta p_k\right]$$

Since:

$$\nabla \Delta p_k = \nabla\left(A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} p_s - \left(A_{k-\frac{1}{2}} + B_{k-\frac{1}{2}} p_s\right)\right) = \nabla \Delta B_k p_s = \Delta B_k \nabla p_s$$

We may write

(UPRS, VPRS) $=$

$$R_d T_v \cos\theta \bigtriangledown \ln p = -\tfrac{R_d \cos\theta (T_v)_k}{\Delta p_k}\left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) B_{k-\frac{1}{2}} \nabla p_s + \alpha_k \Delta B_k \nabla p_s\right] =$$

$$= -\tfrac{R_d \cos\theta (T_v)_k}{\Delta p_k}\left[B_{k-\frac{1}{2}} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) + \alpha_k \Delta B_k\right] p_s \nabla \ln p_s$$

$$= -\tfrac{1}{\Delta p_k}\left[B_{k-\frac{1}{2}} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) + \alpha_k \Delta B_k\right] R_d \left(T_v\right)_k p_s \cos\theta \nabla \ln p_s$$

in the code $\quad \cos\theta \nabla \ln p_s = (\mathrm{DPDLAM}, \mathrm{DPDPHI})$

let $\quad \mathrm{cof}\, b = -\tfrac{1}{\Delta p_k}\left[B_{k-\frac{1}{2}} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) + \alpha_k \Delta B_k\right]$

Then $\quad (\mathrm{UPRS}, \mathrm{VPRS}) = \mathrm{cof}\, b\, R_d \left(T_v\right)_k p_s \ (\mathrm{DPDLAM}, \mathrm{DPDPHI})$

for $\quad k = 1 \quad \ln\left(\frac{p_{\frac{3}{2}}}{p_{\frac{1}{2}}}\right) \nabla p_{\frac{1}{2}}$.

Since $p_{\frac{1}{2}} = 0$, we have to find the limit:

$$\lim_{p_{\frac{1}{2}} \to 0} \left(\ln\left(\frac{p_{\frac{3}{2}}}{p_{\frac{1}{2}}}\right) \nabla p_{\frac{1}{2}}\right) = 0$$

therefore $\quad (\mathrm{UPRS}, \mathrm{VPRS})_{k=1} = -R_d \cos\theta \tfrac{(T_k)_1}{\Delta p_1} \ (\mathrm{DPDLAM}, \mathrm{DPDPHI})$

1.8 Calculation of (uphi, vphi)

(uphi, vphi) $= -\cos\theta \nabla \phi_k$

$$\phi_k = \phi_{k+\frac{1}{2}} + \alpha_k R_d \left(T_v\right)_k \ , \quad \phi_{k+\frac{1}{2}} = \phi_s + \sum_{j=k+1}^{Levs} R_d \left(T_v\right)_j \ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right)$$

$$\alpha_1 = \ln 2 \quad \text{or} \quad (=1) \ , \quad \alpha_k = 1 - \frac{p_{k+\frac{1}{2}}}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \quad k > 1 \ .$$

then: $-\cos\theta\nabla\phi_k = -\cos\theta\nabla\left[\phi_s + \sum_{j=k+1}^{Levs} R_d \left(T_v\right)_j \ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right) + \alpha_k R_d \left(T_v\right)_k\right]$

$$= -\cos\theta\nabla\phi_s - R_d\cos\theta\left\{\sum_{j=k+1}^{Levs}\left[\left(T_v\right)_j \nabla\pi_i + \pi_j\nabla\left(T_v\right)_j\right] + \alpha_k\nabla\left(T_v\right)_k + \left(T_v\right)_k\nabla\alpha_k\right\}$$

where $\pi_j = \ln\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}$

let (uphi, vphi) $= \overrightarrow{px_1} + \overrightarrow{px_2} + \overrightarrow{px_3} + \overrightarrow{px_4} + \overrightarrow{px_5}$ (for coding purposes)

$$\overrightarrow{px_1} = -\cos\theta\nabla\phi_s \ , \quad \overrightarrow{px_2} = -R_d\cos\theta\sum_{j=k+1}^{Levs}\left(T_v\right)_j\nabla\pi_j$$

$$\overrightarrow{px_3} = -R_d\cos\theta\sum_{j=k+1}^{Levs}\pi_j\nabla\left(T_v\right)_j$$

$$\overrightarrow{px_4} = -R_d\cos\theta\alpha_k\nabla\left(T_v\right)_k \ , \quad \overrightarrow{px_5} = -R_d\cos\theta\left(T_v\right)_k\nabla\alpha_k$$

Evaluate $\overrightarrow{px_2}$

$$\overrightarrow{px_2} = -R_d\cos\theta\sum_{j=k+1}^{Levs}\left(T_v\right)_j\nabla\pi_j$$

$$\nabla\pi_j = \nabla\ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right) = \frac{\nabla p_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{\nabla p_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} =$$

$$= \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}}\nabla p_s - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}}\nabla p_s = \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right)p_s\nabla\ln p_s$$

$$\overrightarrow{px_2} = -R_d\cos\theta\sum_{j=k+1}^{Levs}\left(T_v\right)_j\left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right)p_s\nabla\ln p_s$$

$$\overrightarrow{px_2} = -R_d \sum_{j=k+1}^{Levs} (T_v)_j \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right) \; p_s \underbrace{(\mathrm{DPDLAM}, \mathrm{DPDPHI})}_{\cos\theta \nabla \ln p_s}$$

for a pure σ coordinate $\quad A_{k+\frac{1}{2}} = 0 \quad \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} = \frac{1}{p_s} - \frac{1}{p_s} = 0$

$$\overrightarrow{px_2} = 0$$

Evaluate $\overrightarrow{px_3}$

$$\overrightarrow{px_3} = -R_d \cos\theta \sum_{j=k+1}^{Levs} \ln \frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \nabla (T_v)_j$$

In the code $\quad \cos\theta \nabla T = \cos^2\theta \;$ (dtdl, dtdf)

$$\overrightarrow{px_3} = -R_d \cos^2\theta \sum_{j=k+1}^{Levs} \ln \frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \; (\,\mathrm{dtdl(j)},\, \mathrm{dtdf(j)}\,)$$

$$\overrightarrow{px_4} = -R_d \cos\theta \alpha_k \nabla (T_v)_k = -R_d \cos^2\theta \alpha_k \; (\,\mathrm{dtdl(k)},\, \mathrm{dtdf(k)}\,)$$

Evaluate $\overrightarrow{px_5}$

$$\overrightarrow{px_5} = -R_d \cos\theta \, (T_v)_k \, \nabla \alpha_k$$

now:

$$\nabla \alpha_k = \nabla \left(1 - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \right) = -\frac{p_{k-\frac{1}{2}}}{\Delta p_k} \left\{ \nabla \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \right\} - \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \left\{ \nabla \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \right\}$$

evaluate

$$\nabla \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) = \frac{\nabla p_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{\nabla p_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}} = \left(\frac{B_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{B_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) p_s \nabla \ln p_s$$

also:

$$\nabla \frac{p_{k-\frac{1}{2}}}{\Delta p_k} = \frac{B_{k-\frac{1}{2}}}{\Delta p_k} \nabla p_s - p_{k-\frac{1}{2}} \frac{\nabla \Delta p_k}{(\Delta p_k)^2}$$

$$= \frac{B_{k-\frac{1}{2}}}{\Delta p_k} \nabla p_s - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \left[\left(B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} \right) \nabla p_s \right] = \left(\frac{B_{k-\frac{1}{2}}}{\Delta p_k} - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \Delta B_k \right) p_s \nabla \ln p_s$$

then

$$\nabla \alpha_k = -\frac{p_{k-\frac{1}{2}}}{\Delta p_k} \left(\frac{B_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{B_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) p_s \nabla \ln p_s - \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \left(\frac{B_{k-\frac{1}{2}}}{\Delta p_k} - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \Delta B_k \right) p_s \nabla \ln p_s$$

$$\nabla \alpha_k = -\frac{1}{\Delta p_k} \left\{ \frac{p_{k-\frac{1}{2}}}{p_{k+\frac{1}{2}}} B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \left(B_{k-\frac{1}{2}} - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \Delta B_k \right) \right\} p_s \nabla \ln p_s$$

let

$$\text{cofa(k)} = -\frac{1}{\Delta p_k} \left\{ \frac{p_{k-\frac{1}{2}}}{p_{k+\frac{1}{2}}} B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \left(B_{k-\frac{1}{2}} - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \Delta B_k \right) \right\} \quad \text{in code}$$

then:

$$\nabla \alpha_k = \ \text{cofa(k)} \, p_s \nabla \ln p_s$$

and finally:

$$\overrightarrow{px_5} = -R_d \cos\theta \, (T_v)_k \, \nabla \alpha_k = -R_d \cos\theta \, (T_v)_k \ \text{cofa(k)} \, p_s \nabla \ln p_s$$

$$\overrightarrow{px_5} = -R_d \, (T_v)_k \ \text{cofa(k)} \, p_s \ (\,\text{DPDLAM, DPDPHI}\,) \quad \text{in code}$$

$$\begin{cases} \text{for pure sigma} \quad A_k = 0 \\ \text{cofa(k)} = -\frac{1}{\Delta p_k}\left\{\frac{B_{k-\frac{1}{2}}p_s}{B_{k+\frac{1}{2}}p_s}B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln\left(\frac{B_{k+\frac{1}{2}}p_s}{B_{k-\frac{1}{2}}p_s}\right)\left(B_{k-\frac{1}{2}} - \frac{\Delta B_k B_{k-\frac{1}{2}}p_s}{\Delta B_k p_s}\right)\right\} p_s \nabla \ln p_s \\ \text{cofa(k)} = 0 \quad \overrightarrow{px_5} = 0 \end{cases}$$

1.9 Semi-Implicit Time Integration

The momentum equation is:

$$\frac{\partial \vec{v}}{\partial t} = x^* - \frac{\beta}{2}\Delta_{tt}\left(A\nabla T + R_d T_r \nabla Q\right) \quad \Delta x_{tt} = x^+ + x^- - 2x \ , \quad Q = \ln p_s$$

Operating with the divergence operator we have

$$\frac{\partial D}{\partial t} = x - \frac{\beta}{2}\Delta_{tt}\left(A\nabla^2 T + R_d T_r \nabla^2 Q\right)$$

where x^* and x are the appropriate non-linear terms.

We are dealing with the three coupled equations:

$$\begin{cases} \frac{\partial D}{\partial t} = x + \frac{\beta}{2}\frac{n(n+1)}{a^2}\left\{A\left(T^+ + T^- - 2T\right) + R_d T_r \left(Q^+ + Q^- - 2Q\right)\right\} \\ \frac{\partial T}{\partial t} = Y - \frac{\beta}{2}\Delta_{tt} BD = Y - \frac{\beta}{2}B\left(D^+ + D^- - 2D\right) \\ \frac{\partial Q}{\partial t} = Z - \frac{\beta}{2}\Delta_{tt}\left(S \cdot D\right) = Z - \frac{\beta}{2}S \cdot \left(D^+ + D^- - 2D\right) \end{cases}$$

$S \cdot X = \sum_1^{Levs} S_i X_i \ , \quad S_i = \frac{\Delta p_i^r}{p^r} \ , \quad p^r = p$ ref. $= 850$ mb.

Now finite difference in time, solve for D^+ then T^+ , Q^+

$$\begin{cases} D^+ = D^- + 2\Delta t x + \beta\frac{\Delta t n(n+1)}{a^2}\left\{A\left(T^- - 2T\right) + R_d T_r\left(Q^- - 2Q\right)\right\} + \beta\frac{\Delta t n(n+1)}{a^2}\left(AT^+ + R_d T_r Q^+\right) \\ T^+ = T^- + 2\Delta t Y - \beta\Delta t B\left(D^- - 2D\right) - \beta\Delta t B D^+ \\ Q^+ = Q^- + 2\Delta t Z - \beta\Delta t S \cdot \left(D^- - 2D\right) - \beta\Delta t S \cdot D^+ \end{cases}$$

let $\hat{A} = \frac{\beta \Delta t A}{a^2}$ $\hat{T}_r = \frac{\beta \Delta t}{a^2} R T_r$

then

$$D^+ = D^- + 2\Delta t x + n\left(n+1\right)\left\{\hat{A}\left(T^- - 2T\right) + \hat{T}_r\left(Q^- - 2Q\right)\right\} + n\left(n+1\right)\left(\hat{A}T^+ + \hat{T}_r Q^+\right)$$

Let

$$\begin{cases} E_n^l = D^- + 2\Delta t x + n\left(n+1\right)\left\{\hat{A}\left(T^- - 2T\right) + \hat{T}_r\left(Q^- - 2Q\right)\right\} \\ F_n^l = T^- + 2\Delta t Y - \beta \Delta t B\left(D^- - 2D\right) \\ G_n^l = Q^- + 2\Delta t Z - \beta \Delta t S \cdot \left(D^- - 2D\right) \end{cases}$$

then:

$$\begin{cases} D^+ = E_n^l + n\left(n+1\right)\left(\hat{A}T^+ + \hat{T}_r Q^+\right) \\ T^+ = F_n^l - \beta \Delta t B D^+ \\ Q^+ = G_n^l - \beta \Delta t S \cdot D^+ \end{cases} \tag{1.9.1}$$

Solve for D^+

$$D^+ = E_n^l + n\left(n+1\right)\left\{\hat{A}\left(F_n^l - \beta \Delta t B D^+\right) + \hat{T}_r\left(G_n^l - \beta \Delta t S \cdot D^+\right)\right\}$$

collect terms with D^+ :

$$\left\{I + n\left(n+1\right)\beta \Delta t\left(\hat{A}B + T_r S\cdot\right)\right\} D^+ = E_n^l + n\left(n+1\right)\left(\hat{A}F_n^l + \hat{T}_r G_n^l\right)$$

let $D_m = I + n\left(n+1\right)\beta \Delta t\left(\hat{A}B + \hat{T}_r S\right)$

Table 1: Vertical Structure Indexing top to bottom

k	$k+\frac{1}{2}$	ak5	bk5	pk5	$\eta\frac{\partial p}{\partial \eta}$
0	$\frac{1}{2}$	ak5(1)	bk5(1)	pk5(1)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{\frac{1}{2}}$=Dot(1)
1	$1\frac{1}{2}$	ak5(2)	bk5(2)	pk5(2)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{1\frac{1}{2}}$=Dot(2)
2	$2\frac{1}{2}$	ak5(3)	bk5(3)	pk5(3)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{2\frac{1}{2}}$=Dot(3)
3	$3\frac{1}{2}$	ak5(4)	bk5(4)	pk5(4)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{3\frac{1}{2}}$=Dot(4)

Then $\quad D^{+} = D_m^{-1}\left\{E_n^l + n\,(n+1)\left(\hat{A}F_n^l + \hat{T}_r G_n^l\right)\right\}$

The predicted values of T^{+} and Q^{+} are given by eq. (1.9.1)

INDEXING AND LOOPING.

consider $\overrightarrow{px_2}$:

$$\overrightarrow{px_2}\,(k) = -R_d\left[\sum_{j=k+1}^{Levs}(T_v)_j\left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right)\right] p_s \;\;(\,\mathrm{DPDLAM,\ DPDPHI}\,) = (\,\mathrm{px2u,\ px2v}\,)$$

Using the notation in Table 1

$$\text{let}\;\; f_{j+\frac{1}{2}} = \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}}\;,\quad \mathrm{px(k)} = \sum_{j=k+1}^{Levs}(T_v)_j\left(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}\right)\qquad k = 1,\,\ldots,\,Levs$$

Let Levs=4 k=1, 2, 3, 4 $\quad px = \mathrm{px2\ scalar}\,,$ in code

$$\overrightarrow{px_2} = (\,\mathrm{px2u,\ px2v}\,)$$

k=Levs 4	$\mathrm{px}\,(4) = \sum_{j=4+1}^{4} \equiv 0$
k=Levs-1 3	$\mathrm{px}\,(3) = \sum_{j=3+1}^{4} = (T_v)_4 \left(f_{4+\frac{1}{2}} - f_{4-\frac{1}{2}}\right)$
k=Levs-2 2	$\mathrm{px}\,(2) = \sum_{j=2+1}^{4} = \sum_{j=3}^{4} = (T_v)_3 \left(f_{3+\frac{1}{2}} - f_{3-\frac{1}{2}}\right) + (T_v)_4 \left(f_{4+\frac{1}{2}} - f_{4-\frac{1}{2}}\right)$ $= (T_v)_3 \left(f_{3+\frac{1}{2}} - f_{3-\frac{1}{2}}\right) + \mathrm{px}\,(3)$
k=Levs-3 1	$\mathrm{px}\,(1) = \sum_{j=1+1}^{4} = \sum_{j=2}^{4} = (T_v)_2 \left(f_{2+\frac{1}{2}} - f_{2-\frac{1}{2}}\right) + \mathrm{px}\,(2)$

$$
\begin{array}{ll}
 & \left\{ \begin{array}{l}
\mathrm{DO}\ \ \mathrm{k} = 2,\, 4-1 \ \ (\mathrm{k} = 2,\, 3\,) \\[1em]
\quad \mathrm{px}\,(2) = \mathrm{px}\,(3) - R_d \left(\frac{\mathrm{B}(4)}{\mathrm{p}(4)} - \frac{\mathrm{B}(3)}{\mathrm{p}(3)} \qquad \right) \ \mathrm{Tg}\,(3) \\[1em]
\quad \mathrm{px}\,(1) = \mathrm{px}\,(2) - R_d \left(\frac{\mathrm{B}(3)}{\mathrm{p}(3)} - \frac{\mathrm{B}(2)}{\mathrm{p}(2)} \qquad \right) \ \mathrm{Tg}\,(2)
\end{array} \right.
\end{array}
$$

$$\mathrm{k} = 2$$

$$\mathrm{k} = 3$$

$\omega\,(\eta)$ in the physics.

The vertical velocity ω is calculated as in the energy conversion term and is passed on to the physical parameterization routines.

Indexing is assumed top to bottom

$P_{1/2} = \mathrm{PK5}(1)$ ——— 1/2

$P_{1\,1/2} = \mathrm{PK5}(2)$ ——— 1 1/2

$P_{2\,1/2} = \mathrm{PK5}(3)$ ——— 2 1/2

$P_{3\,1/2} = \mathrm{PK5}(4)$ ——— 3 1/2

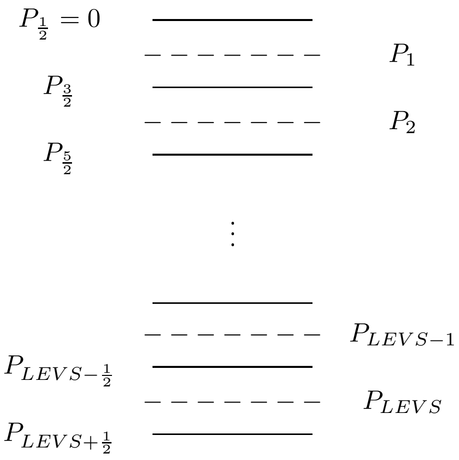
$P_{4\,1/2} = \mathrm{PK5}(5)$ ——— 4 1/2

1.10 Schematic Vertical Grid

LEVS = number of model layers

$P_{K+\frac{1}{2}}$ = pressure at interfaces

P_k = pressure in model layers



$a_{k+\frac{1}{2}}$, $b_{k+\frac{1}{2}}$ are defined at interfaces

1.11 Notation

λ	longitude
θ	latitude
η	hybrid vertical coordinate
A_k B_k	level dependent constants defining η
t	time
a	radius of the earth
f	coriolis parameter
(u, v)	horizontal velocity components
p	pressure
p_s	surface pressure
p_r	reference pressure used in linearization for semi-implicit time integration
$\omega = \frac{\mathrm{d}p}{\mathrm{d}t}$	
T	temperature
T_r	reference temperature
T_v	virtual temperature
ϕ	geopotential
ρ	density
$\alpha = \frac{1}{\rho}$	specific volume
q	specific humidity
Q	energy per unit mass
R_d	dry air gas constant
c_p	dry air specific heat at constant pressure
c_v	dry air specific heat at constant volume
κ	$= R_d/c_p$
ϵ	$= R_v/R_d + 1$
δ	C_{pv}/C_{pd}
Y	hydrostatic matrix

1.12 References

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