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DISCRETE GENERALIZED HYBRID VERTICAL COORDINATES BY A MASS, ENERGY AND ANGULAR MOMENTUM CONSERVING VERTICAL FINITE-DIFFERENCE SCHEME

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ABSTRACT

A detailed discretization of a hydrostatic primitive equation global atmospheric model on spherical and generalized hybrid vertical coordinates is described. The discretization in the horizontal using a spectral method with spherical transformation is not the major theme and is not described in this manuscript. Only the vertical discretization is described in detail up to the level of readiness for programming.

Energy and angular momentum conservation are used as constraints to discretize the vertical integration by finite difference scheme. The entire atmosphere is divided into several layers; only pressure and vertical flux are specified at the interfaces, and other variables such as horizontal wind, temperature, specific humidity and specific amount of tracers are specified at each layer. Conservation is a constraint that requires the pressure at each layer to be averaged by the pressures at the immediate neighbor interfaces (the one above and one below a given layer). Since pressures are not combined from a pressure gradient and density in a logarithmic form, the relationship for pressure between layers and interfaces becomes simple, and with pressure equation not in logarithmic form, it provides mass conservation as an extra.

Due to the generalized vertical coordinate, vertical flux is solved by applying local changes in the pressure and virtual temperature equations to the definition of the vertical coordinate. It solves vertical fluxes at all interfaces by a simple algebraic equation through matrix inversion. For the sake of time splitting between dynamics and physics, the vertical flux obtained in the dynamics is without local changes from model physics, and then the vertical advection is required in the model physics. The semi-implicit time integration scheme is also given by this finite difference scheme in generalized vertical coordinates. The details of the matrixes are described so as to be ready for programming.

A specific definition of a generalized hybrid coordinate, including pressure and isentropic surfaces, is introduced. Due to this definition, pressure is given by surface pressure and virtual temperature. These modifications of the generalized form are necessary to save computational resources. Instead of solving for the pressure at all interfaces, only the surface pressure equation is needed. Though the elements in the matrixes for semi-implicit computations become more complicated than those in the generalized pressure equation at all levels, the computing time is not increased because the matrixes are of the same degree; even two matrixes reduce to vector computation only because the surface pressure is solved instead of pressure at all the interfaces.

1. Introduction

It has become a trend to use generalized vertical coordinates in atmospheric modeling (Simmons and Burridge 1981; Zhu et al 1992; Konor and Arakawa 1997; Johnson and Yuan 1998; Benjamin et al 2004). With generalized coordinates, the atmospheric model can be integrated along different types of coordinate surfaces. The coordinates near the surface and lower atmosphere still use terrain-following sigma coordinates, but over the upper atmosphere better results come from computations on quasi-horizontal coordinates, such as pressure surfaces or isentropic surfaces, that reduce the numerical errors due to miscalculated vertical motions. The combination of these coordinates into a hybrid coordinate system can take advantage of the strengths of the individual types of coordinate surfaces for numerical purposes. The dynamics group in EMC has made an effort to move in this direction as well.

Instead of following what others have done, we plan to have our own system with an incremental implementation. All prognostic variables that we used are included as the prognostic variables in the hybrid vertical coordinate system. After the selection of prognostic variables, we keep using spectral computation in the horizontal and finite difference in the vertical in the first implementation, though we plan to eventually have a semi-Lagrangian, finite or spectral element in vertical. Due to the fact that the hybrid coordinate equation set is different from what we have, a new discretization in the vertical is required to satisfy energy and angular momentum conservation. The matrixes used for the semi-implicit time integration have to be modified due to the different vertical discretization in hybrid coordinates.

This note describes the discretization of a hydrostatic version of a primitive equation global model on spherical and generalized hybrid coordinates. We will still keep a spectral computation in the horizontal, as mentioned, and use finite differencing in the vertical. For backward compatibility, we will use virtual temperature as a model prognostic variable. Section 2 lists the completed set of all continuous equations, and introduces a map factor to rewrite the equation set on spherical coordinates to a regular latitude/longitude pseudo spherical coordinate. Section 3 discusses the vertical constraints in continuous forms that are ready for detailed discretization in Section 4. Section 5 illustrates the process to solve the vertical flux for vertical advections. Section 6 describes the semi-implicit method with the help of linearized equations of divergence, virtual temperature and pressure. A specific definition of the hybrid coordinate is introduced in Section 7, and a discussion of it is in Section 8.

2. Hydrostatic system on spherical and generalized hybrid coordinates

A primitive hydrostatic system on spherical coordinates in the horizontal and a generalized hybrid coordinate in the vertical can be obtained from a textbook, such as Haltiner and Williams (1979) (with a general vertical coordinate transform) and it can be written as

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{a\cos\phi\partial\lambda} - v\frac{\partial u}{a\partial\phi} - \dot{\zeta}\frac{\partial u}{\partial\zeta} - R_d T_v \frac{\partial \ln p}{a\cos\phi\partial\lambda} - g\frac{\partial z}{a\cos\phi\partial\lambda} + f_s v + \frac{uv\tan\phi}{a} + F_u \qquad (2.1a)$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{a\cos\phi\partial\lambda} - v\frac{\partial v}{a\partial\phi} - \dot{\zeta}\frac{\partial v}{\partial\zeta} - R_d T_v \frac{\partial \ln p}{a\partial\phi} - g\frac{\partial z}{a\partial\phi} - f_s u - \frac{u^2\tan\phi}{a} + F_v \qquad (2.1b)$$

$$\frac{\partial T_v}{\partial t} = -u\frac{\partial T_v}{a\cos\phi\partial\lambda} - v\frac{\partial T_v}{a\partial\phi} - \dot{\zeta}\frac{\partial T_v}{\partial\zeta} + \kappa T_v\frac{d\ln p}{dt} + F_{T_v} \tag{2.1c}$$

$$\frac{\partial(\partial p/\partial\zeta)}{\partial t} = -u\frac{\partial(\partial p/\partial\zeta)}{a\cos\phi\partial\lambda} - v\frac{\partial(\partial p/\partial\zeta)}{a\partial\phi} - \dot{\zeta}\frac{\partial(\partial p/\partial\zeta)}{\partial\zeta} - \frac{\partial p}{\partial\zeta}\left[\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial v\cos\phi}{a\cos\phi\partial\phi}\right) + \frac{\partial\dot{\zeta}}{\partial\zeta}\right] \tag{2.1d}$$

$$\frac{\partial q_i}{\partial t} = -u\frac{\partial q_i}{a\cos\phi\partial\lambda} - v\frac{\partial q_i}{a\partial\phi} - \dot{\zeta}\frac{\partial q_i}{\partial\zeta} + F_{q_i} \tag{2.1e}$$

with the hydrostatic relation as

$$\frac{\partial z}{\partial\zeta} = -\frac{R_d T_v}{gp}\frac{\partial p}{\partial\zeta} \tag{2.2}$$

for connecting between the vertical pressure gradient and height changes with coordinates,

$$\kappa = \frac{R_d}{C_p} = \frac{R_d}{C_{pd} + (C_{pv} - C_{pd})q} = \frac{R_d}{C_{pd}}\frac{1}{\left(1 + (C_{pv}/C_{pd} - 1)q\right)} = \frac{\kappa_d}{1 + (C_{pv}/C_{pd} - 1)q} \tag{2.3}$$

includes the contribution of moist air as used in ECMWF model (Untch and Hortel, 2003); u and v are horizontal winds, $\dot{\zeta}$ is vertical coordinate velocity, Tv is virtual temperature, p is pressure, q with index i are tracers including specific humidity. The F with related suffix, the last term in each equation, is the parameterization of model physics. The horizontal coordinates λ and ϕ are spherical longitude and latitude. The rest are traditionally used, for examples g as gravitational force, a as earth radius, fs as sine component of Coriolis force, and z as height from surface.

In order to provide easy reading and easier coding, the concept of a mapping factor to map spherical coordinates in a Gaussian grid to equal latitude and longitude grids is introduced here. Thus, the equation set above can be re-written into a different form of derivative in the horizontal as

$$\frac{\partial u^*}{\partial t} = -m^2u^*\frac{\partial u^*}{a\partial\lambda} - m^2v^*\frac{\partial u^*}{a\partial\varphi} - \dot{\zeta}\frac{\partial u^*}{\partial\zeta} - R_d\frac{T_v}{p}\frac{\partial p}{a\partial\lambda} - g\frac{\partial z}{a\partial\lambda} + f_sv^* + F_u^* \tag{2.4a}$$

$$\frac{\partial v^*}{\partial t} = -m^2u^*\frac{\partial v^*}{a\partial\lambda} - m^2v^*\frac{\partial v^*}{a\partial\varphi} - \dot{\zeta}\frac{\partial v^*}{\partial\zeta} - R_d\frac{T_v}{p}\frac{\partial p}{a\partial\varphi} - g\frac{\partial z}{a\partial\varphi} - f_su^* - m^2\frac{s^{*2}}{a}\sin\phi + F_v^* \tag{2.4b}$$

$$\frac{\partial T_v}{\partial t} = -m^2u^*\frac{\partial T_v}{a\partial\lambda} - m^2v^*\frac{\partial T_v}{a\partial\varphi} - \dot{\zeta}\frac{\partial T_v}{\partial\zeta} + \kappa\frac{T_v}{p}\frac{dp}{dt} + F_{T_m} \tag{2.4c}$$

$$\frac{\partial(\partial p/\partial\zeta)}{\partial t} = -m^2\frac{\partial u^*(\partial p/\partial\zeta)}{a\partial\lambda} - m^2\frac{\partial v^*(\partial p/\partial\zeta)}{a\partial\varphi} - \frac{\partial\dot{\zeta}(\partial p/\partial\zeta)}{\partial\zeta} \tag{2.4d}$$

$$\frac{\partial q_i}{\partial t} = -m^2u^*\frac{\partial q_i}{a\partial\lambda} - m^2v^*\frac{\partial q_i}{a\partial\varphi} - \dot{\zeta}\frac{\partial q_i}{\partial\zeta} + F_{q_i} \tag{2.4e}$$

where

$$m = \frac{1}{\cos\phi} \tag{2.5}$$

$$\Delta\varphi = m\Delta\phi = \frac{1}{\cos\phi}\Delta\phi \tag{2.6}$$

In this case, true spherical λ and ϕ, longitude and latitude, coordinates are mapped into pseudo-spherical coordinates by λ and φ. Thus, the derivatives in the spherical coordinate grid become as simple as

$$\frac{\partial A}{\partial x} = \frac{\partial A}{a\partial\lambda} \tag{2.7}$$

$$\frac{\partial A}{\partial y} = \frac{\partial A}{a\partial\varphi} \tag{2.8}$$

It can be viewed as a Cartesian grid system with mapping factor m. Then pseudo winds are used to simplify the equation with a mapping factor as

$$u^* = u\cos\phi = \frac{u}{m} \tag{2.9}$$

$$v^* = v\cos\phi = \frac{v}{m} \tag{2.10}$$

$$s^{*2} = u^{*2} + v^{*2} \tag{2.11}$$

Note that there is no curvature term in Eq. (2.4a) as compared to Eq. (2.1a), and there is a different form of curvature term shown in Eq. (2.4b) as compared to Eq. (2.1b). Those curvature terms in Eq. (2.1) are combined with horizontal advection with pseudo wind to form the results in Eq. (2.4).

Furthermore, Eq. (2.1) will be used to derive energy conservation in Section 3 because it is a straightforward derivation, and Eq. (2.4) will be used for discretization, so it can be used to code the model directly. Note that this equation set can be solved as a closed system, except for hybrid vertical velocity, which will be discussed later in Section 5.

3. Constraints for Vertical Discretization

In this section, we will describe the continuous form of constraints proposed by Arakawa and Lamb (1977) in designing a vertical differencing scheme. Two constraints are followed:

(1) Vertical integrated pressure gradient force generates no circulation along a contour of surface topography.

We can express this condition by the following derivation:

$$\int_{\zeta_s}^{\zeta_T} \frac{\partial p}{\partial \zeta}(\nabla\Phi + \frac{R_d T_v}{p}\nabla p)d\zeta = \int_{\zeta_s}^{\zeta_T} [\nabla\left(\frac{\partial p}{\partial \zeta}\Phi\right) - \Phi\nabla\frac{\partial p}{\partial \zeta} + \frac{\partial p}{\partial \zeta}\frac{R_d T_v}{p}\nabla p]d\zeta$$
$$= \int_{\zeta_s}^{\zeta_T} \nabla\left(\frac{\partial p}{\partial \zeta}\Phi\right)d\zeta - \int_{\zeta_s}^{\zeta_T}(\Phi\nabla\frac{\partial p}{\partial \zeta} + \frac{\partial \Phi}{\partial \zeta}\nabla p)d\zeta$$
$$= \nabla\int_{\zeta_s}^{\zeta_T}\left(\frac{\partial p}{\partial \zeta}\Phi\right)d\zeta - \int_{\zeta_s}^{\zeta_T}\frac{\partial \Phi\nabla p}{\partial \zeta}d\zeta \qquad (3.1)$$
$$= \nabla\int_{\zeta_s}^{\zeta_T}\left(\frac{\partial p}{\partial \zeta}\Phi\right)d\zeta - \Phi_T\nabla p_T + \Phi_s\nabla p_s$$

where the suffixes s and T denote the surface and top of the model atmosphere. If we assume the top of atmosphere to be a material surface of constant pressure (it is zero in our current model), then the second term on the right hand side of the final form of Eq. (3.1) is zero. Then we apply horizontal integration over the globe above the surface topography, making the first term zero; the third term is also zero along the surface globally, in the final form of Eq. (3.1). This condition is angular momentum conservation, which is used by Simmon and Burridge (1981), Konor and Arakawa (1997), and others.

(2) The energy conservation terms in the thermodynamics and kinetic energy equations have the same form with opposite signs, so that total energy is conserved under adiabatic frictionless conditions.

We can express this condition with thermodynamic energy and kinetic energy. Let's start from thermodynamic equation and continuity equation as

$$\frac{\partial p}{\partial \zeta}C_p\left[\frac{\partial T_v}{\partial t} = -u\frac{\partial T_v}{a\cos\phi\partial\lambda} - v\frac{\partial T_v}{a\partial\phi} - \dot{\zeta}\frac{\partial T_v}{\partial \zeta} + \frac{\kappa T_v}{p}\frac{dp}{dt}\right] \qquad (3.2)$$

$$C_p T_v\left[\frac{\partial(\partial p/\partial\zeta)}{\partial t} = -\left(\frac{\partial u(\partial p/\partial\zeta)}{a\cos\phi\partial\lambda} + \frac{\partial v\cos\phi(\partial p/\partial\zeta)}{a\cos\phi\partial\phi}\right) - \frac{\partial\dot{\zeta}(\partial p/\partial\zeta)}{\partial\zeta}\right] \qquad (3.3)$$

combining Eqs. (3.2) and (3.3), we have

$$\frac{\partial(\frac{\partial p}{\partial\zeta})C_pT_v}{\partial t} = -\left(\frac{\partial u(\frac{\partial p}{\partial\zeta})C_pT_v}{a\cos\phi\partial\lambda} + \frac{\partial v\cos\phi(\frac{\partial p}{\partial\zeta})C_pT_v}{a\cos\phi\partial\phi}\right) - \frac{\partial\dot{\zeta}(\frac{\partial p}{\partial\zeta})C_pT_v}{\partial\zeta} + \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\frac{dp}{dt} \qquad (3.4)$$

where the last term in Eq. (3.4) is called the energy conversion term. Next, let's derive from horizontal momentum equation and continuity equation the following:

$$\frac{\partial p}{\partial \zeta}u\left[\frac{\partial u}{\partial t} = -u\frac{\partial u}{a\cos\phi\partial\lambda} - v\frac{\partial u}{a\partial\phi} - \dot{\zeta}\frac{\partial u}{\partial\zeta} - \frac{R_dT_v}{p}\frac{\partial p}{a\cos\phi\partial\lambda} - \frac{\partial\Phi}{a\cos\phi\partial\lambda} + f_sv + \frac{uv\tan\phi}{a}\right] \qquad (3.5)$$

$$\frac{\partial p}{\partial \zeta}v\left[\frac{\partial v}{\partial t} = -u\frac{\partial v}{a\cos\phi\partial\lambda} - v\frac{\partial v}{a\partial\phi} - \dot{\zeta}\frac{\partial v}{\partial\zeta} - \frac{R_dT_v}{p}\frac{\partial p}{a\partial\phi} - \frac{\partial\Phi}{a\partial\phi} - f_s u - \frac{u^2\tan\phi}{a}\right] \tag{3.6}$$

$$\frac{1}{2}(u^2+v^2)\left[\frac{\partial(\partial p/\partial\zeta)}{\partial t} = -\left(\frac{\partial u(\partial p/\partial\zeta)}{a\cos\phi\partial\lambda} + \frac{\partial v\cos\phi(\partial p/\partial\zeta)}{a\cos\phi\partial\phi}\right) - \frac{\partial\dot{\zeta}(\partial p/\partial\zeta)}{\partial\zeta}\right] \tag{3.7}$$

combining the three equations in Eqs. (3.5), (3.6) and (3.7) above, we obtain

$$\frac{\partial\frac{\partial p}{\partial\zeta}K}{\partial t} + \nabla\bullet V\frac{\partial p}{\partial\zeta}K + \frac{\partial\dot{\zeta}\frac{\partial p}{\partial\zeta}K}{\partial\zeta} = -\frac{\partial p}{\partial\zeta}\vec{V}\bullet\left(\nabla\Phi + \frac{R_dT_v}{p}\nabla p\right) \tag{3.8}$$

where K is kinetic energy as $(u^2+v^2)/2$ and Eq. (3.8) can be expanded, by applying the continuity equation and total derivative equation, as

$$\begin{aligned}
-\frac{\partial p}{\partial\zeta}\vec{V}\bullet\left(\nabla\Phi + \frac{R_dT_v}{p}\nabla p\right) &= -\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) + \Phi\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\vec{V}\right) - \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\vec{V}\bullet\nabla p \\
&= -\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) - \Phi\left(\frac{\partial(\frac{\partial p}{\partial\zeta})}{\partial t} + \frac{\partial\dot{\zeta}(\frac{\partial p}{\partial\zeta})}{\partial\zeta}\right) - \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\vec{V}\bullet\nabla p \\
&= -\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) - \Phi\left(\frac{\partial(\frac{\partial p}{\partial\zeta})}{\partial t} + \frac{\partial\dot{\zeta}(\frac{\partial p}{\partial\zeta})}{\partial\zeta}\right) - \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\left(\frac{dp}{dt} - \frac{\partial p}{\partial t} - \dot{\zeta}\frac{\partial p}{\partial\zeta}\right) \\
&= -\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) - \Phi\left(\frac{\partial(\frac{\partial p}{\partial t})}{\partial\zeta} + \frac{\partial\dot{\zeta}(\frac{\partial p}{\partial\zeta})}{\partial\zeta}\right) - \frac{\partial\Phi}{\partial\zeta}\left(\frac{\partial p}{\partial t} + \dot{\zeta}\frac{\partial p}{\partial\zeta}\right) - \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\frac{dp}{dt} \\
&= -\nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) - \frac{\partial}{\partial\zeta}\left[\Phi\left(\frac{\partial p}{\partial t} + \dot{\zeta}\frac{\partial p}{\partial\zeta}\right)\right] - \frac{\partial p}{\partial\zeta}\frac{R_dT_v}{p}\frac{dp}{dt}
\end{aligned} \tag{3.9}$$

Combining Eqs. (3.4) and (3.8) with the expanded of Eq. (3.8) in the final form of Eq. (3.9), we obtain total energy as

$$\frac{\partial\frac{\partial p}{\partial\zeta}E}{\partial t} = -\nabla\bullet V\frac{\partial p}{\partial\zeta}E - \frac{\partial\dot{\zeta}\frac{\partial p}{\partial\zeta}E}{\partial\zeta} - \nabla\bullet\left(\frac{\partial p}{\partial\zeta}\Phi\vec{V}\right) - \frac{\partial}{\partial\zeta}\left[\Phi\left(\frac{\partial p}{\partial t} + \dot{\zeta}\frac{\partial p}{\partial\zeta}\right)\right] \tag{3.10}$$

where E=CpTv+K. All terms on the right hand side of Eq. (3.10) will be zero upon global integration in the horizontal and in vertical. Thus, it implies that total energy is conserved. The last term in Eqs (3.4) and (3.9), called the energy conversion term, as mentioned above, is the same but of opposite sign between the thermodynamic and kinetic energy. If the discretization used in Eq. (3.9) is used for Eq. (3.4), in other words, if the discretization used for the energy conversion term is the same between the momentum equation and thermodynamic equation, then the total energy will be conserved.

4. Vertical Discretization

This section uses the constraints described in Section 3 to resolve the discretization for all equations. First, we decide the vertical grid structure, then use the grid structure and the constraints in the previous section to do the discretization. The vertical grid structure is shown in Fig.1. Only pressure and vertical flux are located at the interfaces (or levels), all other variables are located with layers. Hereafter, a variable with a hat indicates that it is located at the interfaces, otherwise the variables are located with layers.

From the vertical grid structure in Fig. 1, we can start to derive the first constraint from a discretization of the first form in Eq. (3.1) and end up with a discretization of the last form in Eq. (3.1) as the following:

$$\begin{aligned}\sum_{k=1}^{K}\left(\frac{\partial p}{\partial \zeta}\right)_k (\nabla\Phi + \frac{R_d T_v}{p}\nabla p)_k \Delta\zeta_k &= \sum_{k=1}^{K}[\nabla\left(\frac{\partial p}{\partial \zeta}\Phi\right) - \Phi\nabla\frac{\partial p}{\partial \zeta} + \frac{\partial p}{\partial \zeta}\frac{R_d T_v}{p}\nabla p]_k \Delta\zeta_k \\ &= \sum_{k=1}^{K}\nabla\left(\frac{\partial p}{\partial \zeta}\Phi\right)_k \Delta\zeta_k + \sum_{k=1}^{K}[-\Phi_k\nabla\Delta p_k + \Delta p_k\left(\frac{R_d T_v}{p}\right)_k \nabla p_k] \\ &= \nabla\sum_{k=1}^{K}\left(\frac{\partial p}{\partial \zeta}\Phi\right)_k \Delta\zeta_k + \sum_{k=1}^{K}[-\Phi_k\nabla(\hat{p}_{k+1} - \hat{p}_k) + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \nabla p_k] \\ &= \nabla\sum_{k=1}^{K}\left(\frac{\partial p}{\partial \zeta}\Phi\right)_k \Delta\zeta_k - \Phi_T\nabla p_T + \Phi_s\nabla p_s\end{aligned} \tag{4.1}$$

The immediate problem, seen in the third form in Eq. (4.1), is the pressure at a layer, since we provide pressure at levels as shown in Fig. 1. To represent p at layers by p at levels, let $p_k = f(\hat{p}_{k+1}, \hat{p}_k)$ so

$$\nabla p_k = \frac{\partial p_k}{\partial \hat{p}_{k+1}}\nabla\hat{p}_{k+1} + \frac{\partial p_k}{\partial \hat{p}_k}\nabla\hat{p}_k \tag{4.2}$$

for k=1,K. Substitute Eq. (4.2) into the third form in Eq. (4.1), which equals to the last form in Eq. (4.1), then regrouping in terms of pressure gradient at different levels, we obtain

$$\begin{aligned}&\sum_{k=1}^{K}\left[\left\langle -\Phi_k + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_{k+1}}\right\rangle\nabla\hat{p}_{k+1} + \left\langle \Phi_k + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_k}\right\rangle\nabla\hat{p}_k\right] \\ &= -\Phi_T\nabla p_T + \Phi_s\nabla p_s\end{aligned} \tag{4.3}$$

After expanding the above equation in the following:

$$
\begin{aligned}
&\left\langle -\Phi_1 + (\hat{p}_2 - \hat{p}_1)\left(\frac{R_d T_v}{p}\right)_1 \frac{\partial p_1}{\partial \hat{p}_2}\right\rangle \nabla \hat{p}_2 + \left\langle \Phi_1 + (\hat{p}_2 - \hat{p}_1)\left(\frac{R_d T_v}{p}\right)_1 \frac{\partial p_1}{\partial \hat{p}_1}\right\rangle \nabla \hat{p}_1 + \\
&\left\langle -\Phi_2 + (\hat{p}_3 - \hat{p}_2)\left(\frac{R_d T_v}{p}\right)_2 \frac{\partial p_2}{\partial \hat{p}_3}\right\rangle \nabla \hat{p}_3 + \left\langle \Phi_2 + (\hat{p}_3 - \hat{p}_2)\left(\frac{R_d T_v}{p}\right)_2 \frac{\partial p_2}{\partial \hat{p}_2}\right\rangle \nabla \hat{p}_2 + \\
&\cdots\cdots\cdots \\
&\left\langle -\Phi_k + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_{k+1}}\right\rangle \nabla \hat{p}_{k+1} + \left\langle \Phi_k + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_k}\right\rangle \nabla \hat{p}_k + \\
&\cdots\cdots\cdots \\
&\left\langle -\Phi_K + (\hat{p}_{K+1} - \hat{p}_K)\left(\frac{R_d T_v}{p}\right)_K \frac{\partial p_K}{\partial \hat{p}_{K+1}}\right\rangle \nabla \hat{p}_{K+1} + \left\langle \Phi_K + (\hat{p}_{K+1} - \hat{p}_K)\left(\frac{R_d T_v}{p}\right)_K \frac{\partial p_K}{\partial \hat{p}_K}\right\rangle \nabla \hat{p}_K \\
&= -\Phi_T \nabla p_T + \Phi_s \nabla p_s = -\Phi_T \nabla \hat{p}_{K+1} + \Phi_s \nabla \hat{p}_1
\end{aligned}
\tag{4.4}
$$

we have following conditions to satisfy Eq. (4.4) above by

$$
-\Phi_k + (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_{k+1}} + \Phi_{k+1} + (\hat{p}_{k+2} - \hat{p}_{k+1})\left(\frac{R_d T_v}{p}\right)_{k+1} \frac{\partial p_{k+1}}{\partial \hat{p}_{k+1}} = 0 \tag{4.5}
$$

for k=1,K-1, and

$$
\Phi_1 + (\hat{p}_2 - p_s)\left(\frac{R_d T_v}{p}\right)_1 \frac{\partial p_1}{\partial p_s} = \Phi_s \tag{4.6a}
$$

$$
-\Phi_K + (\hat{p}_{K+1} - \hat{p}_K)\left(\frac{R_d T_v}{p}\right)_K \frac{\partial p_K}{\partial \hat{p}_{K+1}} = -\Phi_T \tag{4.6b}
$$

without knowing the values of pressure gradients at all layers. Re-arrange Eq. (4.5), and we have the hydrostatic relation between layers as

$$
\Phi_{k+1} - \Phi_k = -(\hat{p}_{k+2} - \hat{p}_{k+1})\left(\frac{R_d T_v}{p}\right)_{k+1} \frac{\partial p_{k+1}}{\partial \hat{p}_{k+1}} - (\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_{k+1}} \tag{4.7}
$$

for k=1,K-1. Add and subtract the height at level k+1 on the left hand side of Eq. (4.7), regroup it, and rearrange the index, and we have hydrostatic relation between layer and levels as

$$
\Phi_k - \hat{\Phi}_k = -(\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_k} \tag{4.8a}
$$

$$
\hat{\Phi}_{k+1} - \Phi_k = -(\hat{p}_{k+1} - \hat{p}_k)\left(\frac{R_d T_v}{p}\right)_k \frac{\partial p_k}{\partial \hat{p}_{k+1}} \tag{4.8b}
$$

for k=1,K, which also satisfies the relations in Eq. (4.6) for the surface and top of the atmosphere. But the function of p for the layers has not been solved, which can be done by applying the second constraint.

Next, let's use the second and the last version of Eq. (3.9) to determine the energy conversion term from the kinetic equation, then use it for the thermodynamic equation later. Thus, the conversion term can be discretized from the second and last forms of Eq. (3.9) as

$$\left(\frac{\partial p}{\partial \zeta}\frac{R_d T_v}{p}\frac{dp}{dt}\right)_k = -\frac{\partial}{\partial \zeta}\left[\Phi\left(\frac{\partial p}{\partial t}+\dot{\zeta}\frac{\partial p}{\partial \zeta}\right)\right]_k + \Phi_k\left(\frac{\partial \frac{\partial p}{\partial t}}{\partial \zeta}+\frac{\partial \dot{\zeta}\frac{\partial p}{\partial \zeta}}{\partial \zeta}\right)_k + \left(\frac{\partial p}{\partial \zeta}\frac{R_d T_v}{p}\right)_k \vec{V}_k \bullet \nabla p_k$$

$$= \frac{1}{\Delta\zeta_k}\left\langle\left(\Phi_k - \hat{\Phi}_{k+1}\right)\left(\frac{\partial \hat{p}}{\partial t}+\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_{k+1} + \left(\hat{\Phi}_k - \Phi_k\right)\left(\frac{\partial \hat{p}}{\partial t}+\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_k\right\rangle + \left(\frac{\partial p}{\partial \zeta}\frac{R_d T_v}{p}\right)_k \vec{V}_k \bullet \nabla p_k \tag{4.9}$$

for k=1,K. Next, substitute the hydrostatic relation between layers and levels, Eq. (4.8), into Eq. (4.9) and use the discretization of vertical pressure gradient as follows:

$$\left(\frac{\partial p}{\partial \zeta}\right)_k = \frac{\hat{p}_{k+1}-\hat{p}_k}{\Delta\zeta_k} \tag{4.10}$$

for k=1, K, we obtain

$$\left(\frac{dp}{dt}\right)_k = \frac{\partial p_k}{\partial \hat{p}_{k+1}}\left(\frac{\partial \hat{p}}{\partial t}+\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_{k+1} + \frac{\partial p_k}{\partial \hat{p}_k}\left(\frac{\partial \hat{p}}{\partial t}+\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_k + \vec{V}_k \bullet \nabla p_k \tag{4.11}$$

for k=1,K. It is easy to prove that

$$V \bullet \nabla p = u\frac{\partial p}{a\cos\phi\partial\lambda} + v\frac{\partial p}{a\partial\phi} = m^2 u^* \frac{\partial p}{a\partial\lambda} + m^2 v^* \frac{\partial p}{a\partial\varphi} = m^2 V^* \bullet \nabla p \tag{4.12}$$

thus, above equation becomes

$$\left(\frac{dp}{dt}\right)_k = \frac{\partial p_k}{\partial \hat{p}_{k+1}}\frac{\partial \hat{p}_{k+1}}{\partial t} + \frac{\partial p_k}{\partial \hat{p}_k}\frac{\partial \hat{p}_k}{\partial t} + m^2\vec{V}_k^* \bullet \nabla p_k + \frac{\partial p_k}{\partial \hat{p}_{k+1}}\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_{k+1} + \frac{\partial p_k}{\partial \hat{p}_k}\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial \zeta}\right)_k \tag{4.13}$$

for k=1,K, which is the same form as used in Eq. (2.4) with a map factor, and it can be discretized from the above equation by using pressure at layers, as

$$\left(\frac{dp}{dt}\right)_k = \frac{\partial p_k}{\partial t} + m^2\vec{V}_k^* \bullet \nabla p_k + \left(\dot{\zeta}\frac{\partial p}{\partial \zeta}\right)_k \tag{4.14}$$

for k=1,K. Observing the equations above, Eqs. (4.13) and (4.14), we can give a simple solution to the function of p at layers as

$$\frac{\partial p_k}{\partial \hat{p}_{k+1}} = \frac{\partial p_k}{\partial \hat{p}_k} = \frac{1}{2} \tag{4.15}$$

for k=1,K. Therefore, from Eqs. (4.15) and (4.2), we get

$$p_k = \frac{1}{2}(\hat{p}_{k+1} + \hat{p}_k) \tag{4.16}$$

for k=1,K, and the hydrostatic relation in Eq. (4.8) becomes

$$\begin{aligned} \Phi_k - \hat{\Phi}_k &= -R_d \frac{\hat{p}_{k+1} - \hat{p}_k}{\hat{p}_{k+1} + \hat{p}_k}(T_v)_k \\ \hat{\Phi}_{k+1} - \Phi_k &= -R_d \frac{\hat{p}_{k+1} - \hat{p}_k}{\hat{p}_{k+1} + \hat{p}_k}(T_v)_k \end{aligned} \tag{4.17}$$

for k=1,K, so all pressures use pressures at all levels. Expanding Eq. (4.17) from the surface to given layer k, and summing them all, we have

$$\Phi_k = \Phi_s + R_d \sum_{i=1}^{k-1} \frac{\hat{p}_i - \hat{p}_{i+1}}{\hat{p}_i + \hat{p}_{i+1}}(T_v)_i + R_d \sum_{i=1}^{k} \frac{\hat{p}_i - \hat{p}_{i+1}}{\hat{p}_i + \hat{p}_{i+1}}(T_v)_i \tag{4.18}$$

for k=1,K, thus, the pressure gradient force can be written as

$$\begin{aligned} \left(\frac{R_d T_v}{p}\nabla p + \nabla\Phi\right)_k &= R_d (T_v)_k \frac{\nabla\hat{p}_{k+1} + \nabla\hat{p}_k}{\hat{p}_{k+1} + \hat{p}_k} + (\nabla\Phi)_k \\ &= R_d (T_v)_k \frac{\nabla\hat{p}_{k+1} + \nabla\hat{p}_k}{\hat{p}_{k+1} + \hat{p}_k} + \nabla\Phi_s \\ &+ R_d \sum_{i=1}^{k} \left[\frac{\hat{p}_i - \hat{p}_{i+1}}{\hat{p}_i + \hat{p}_{i+1}} \nabla(T_v)_i + \frac{(T_v)_i}{\hat{p}_i + \hat{p}_{i+1}} \nabla(\hat{p}_i - \hat{p}_{i+1}) - \frac{(\hat{p}_i - \hat{p}_{i+1})(T_v)_i}{(\hat{p}_i + \hat{p}_{i+1})^2} \nabla(\hat{p}_i + \hat{p}_{i+1}) \right] \\ &+ R_d \sum_{i=1}^{k-1} \left[\frac{\hat{p}_i - \hat{p}_{i+1}}{\hat{p}_i + \hat{p}_{i+1}} \nabla(T_v)_i + \frac{(T_v)_i}{\hat{p}_i + \hat{p}_{i+1}} \nabla(\hat{p}_i - \hat{p}_{i+1}) - \frac{(\hat{p}_i - \hat{p}_{i+1})(T_v)_i}{(\hat{p}_i + \hat{p}_{i+1})^2} \nabla(\hat{p}_i + \hat{p}_{i+1}) \right] \end{aligned} \tag{4.19}$$

Next, let's discretize the continuity equation by using Eqs (2.4d) and (4.10); we have

$$\begin{aligned} \frac{\partial(\hat{p}_{k+1} - \hat{p}_k)}{\partial t} &= -m^2\left((\hat{p}_{k+1} - \hat{p}_k)\left(\frac{\partial u_k^*}{a\partial\lambda} + \frac{\partial v_k^*}{a\partial\varphi}\right) + u_k^* \frac{\partial(\hat{p}_{k+1} - \hat{p}_k)}{a\partial\lambda} + v_k^* \frac{\partial(\hat{p}_{k+1} - \hat{p}_k)}{a\partial\varphi} \right) \\ &- \left\langle \left(\dot{\zeta}\frac{\partial p}{\partial\zeta}\right)^{\wedge}_{k+1} - \left(\dot{\zeta}\frac{\partial p}{\partial\zeta}\right)^{\wedge}_{k} \right\rangle \end{aligned} \tag{4.20}$$

for k=1,K, then, list the above equation for layers from k to K, the top of the atmosphere

$$\frac{\partial \hat{p}_K}{\partial t} = \frac{\partial \hat{p}_{K+1}}{\partial t} + m^2\left((\hat{p}_{K+1} - \hat{p}_K)\left(\frac{\partial u_K^*}{a\partial\lambda} + \frac{\partial v_K^*}{a\partial\varphi}\right) + u_K^* \frac{\partial(\hat{p}_{K+1} - \hat{p}_K)}{a\partial\lambda} + v_K^* \frac{\partial(\hat{p}_{K+1} - \hat{p}_K)}{a\partial\varphi}\right) + \left\langle \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{K+1} - \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_K \right\rangle$$

$$\frac{\partial \hat{p}_{K-1}}{\partial t} = \frac{\partial \hat{p}_K}{\partial t} + m^2\left((\hat{p}_K - \hat{p}_{K-1})\left(\frac{\partial u_{K-1}^*}{a\partial\lambda} + \frac{\partial v_{K-1}^*}{a\partial\varphi}\right) + u_{K-1}^* \frac{\partial(\hat{p}_K - \hat{p}_{K-1})}{a\partial\lambda} + v_{K-1}^* \frac{\partial(\hat{p}_K - \hat{p}_{K-1})}{a\partial\varphi}\right) + \left\langle \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_K - \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{K-1} \right\rangle$$

....

$$\frac{\partial \hat{p}_{k+1}}{\partial t} = \frac{\partial \hat{p}_{k+2}}{\partial t} + m^2\left((\hat{p}_{k+2} - \hat{p}_{k+1})\left(\frac{\partial u_{k+1}^*}{a\partial\lambda} + \frac{\partial v_{k+1}^*}{a\partial\varphi}\right) + u_{k+1}^* \frac{\partial(\hat{p}_{k+2} - \hat{p}_{k+1})}{a\partial\lambda} + v_{k+1}^* \frac{\partial(\hat{p}_{k+2} - \hat{p}_{k+1})}{a\partial\varphi}\right) + \left\langle \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{k+2} - \left(\dot{\zeta}\frac{\hat{\partial p}_{k+1}}{\partial\zeta}\right)_{k+1} \right\rangle$$

$$\frac{\partial \hat{p}_k}{\partial t} = \frac{\partial \hat{p}_{k+1}}{\partial t} + m^2\left((\hat{p}_{k+1} - \hat{p}_k)\left(\frac{\partial u_k^*}{a\partial\lambda} + \frac{\partial v_k^*}{a\partial\varphi}\right) + u_k^* \frac{\partial(\hat{p}_{k+1} - \hat{p}_k)}{a\partial\lambda} + v_k^* \frac{\partial(\hat{p}_{k+1} - \hat{p}_k)}{a\partial\varphi}\right) + \left\langle \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{k+1} - \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_k \right\rangle$$

then sum them together, after eliminating the same local pressure change, we then obtain the pressure equation for all levels, including surface pressure, as

$$\begin{aligned}\frac{\partial \hat{p}_k}{\partial t} &= -m^2\sum_{i=k}^{K}\left((\hat{p}_i - \hat{p}_{i+1})\left(\frac{\partial u_i^*}{a\partial\lambda} + \frac{\partial v_i^*}{a\partial\varphi}\right) + u_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\lambda} + v_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\varphi}\right) - \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_k \\ &= -m^2\sum_{i=k}^{K}\left((\hat{p}_i - \hat{p}_{i+1})D_i^* + V_i^* \bullet \nabla(\hat{p}_i - \hat{p}_{i+1})\right) - \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_k\end{aligned} \tag{4.21}$$

for k=1,K+1, under the following boundary conditions:

$$\left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{K+1} = \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_1 = 0 \tag{4.22}$$

Note that the top of the atmosphere at model level is treated as a material surface at zero pressure, and not only its value but also it derivative are zero, thus we obtain Eq. (4.21) with the pressure tendency at the top level.

Next let's discretize the thermodynamic equation. First, by substituting Eq. (4.15) into Eq. (4.13), then applying it to the thermodynamic equation, we have

$$\frac{dT_{vk}}{dt} = \frac{\kappa T_{vk}}{\hat{p}_k + \hat{p}_{k+1}}\left[\left(\frac{\partial \hat{p}_k}{\partial t} + \frac{\partial \hat{p}_{k+1}}{\partial t}\right) + m^2\vec{V}_k \bullet \nabla(\hat{p}_k + \hat{p}_{k+1}) + \left\langle \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_k + \left(\dot{\zeta}\frac{\hat{\partial p}}{\partial\zeta}\right)_{k+1} \right\rangle\right] \tag{4.23}$$

for k=1,K. Next, by substituting Eq. (4.21) for levels k and k+1 into Eq. (4.23), the final form of thermodynamic equation, for k=1,K, becomes

$$\frac{dT_{vk}}{dt} = \frac{\kappa T_{vk}}{\hat{p}_k + \hat{p}_{k+1}} \frac{dp}{dt}$$

$$= \frac{\kappa T_{vk}}{\hat{p}_k + \hat{p}_{k+1}} m^2 \Big\langle V_k^* \bullet \nabla(\hat{p}_k + \hat{p}_{k+1}) \tag{4.24a}$$

$$- \sum_{i=k}^{K} \big((\hat{p}_i - \hat{p}_{i+1}) D_i^* + V_i^* \bullet \nabla(\hat{p}_i - \hat{p}_{i+1}) \big) - \sum_{i=k+1}^{K} \big((\hat{p}_i - \hat{p}_{i+1}) D_i^* + V_i^* \bullet \nabla(\hat{p}_i - \hat{p}_{i+1}) \big) \Big\rangle$$

therefore, the omega equation can be written as

$$\omega = m^2 \left[V_k^* \bullet \nabla(\hat{p}_k + \hat{p}_{k+1}) - \sum_{i=k}^{K} \big((\hat{p}_i - \hat{p}_{i+1}) D_i^* + V_i^* \bullet \nabla(\hat{p}_i - \hat{p}_{i+1}) \big) - \sum_{i=k+1}^{K} \big((\hat{p}_i - \hat{p}_{i+1}) D_i^* + V_i^* \bullet \nabla(\hat{p}_i - \hat{p}_{i+1}) \big) \right] \tag{4.24b}$$

Note that the summation is zero while the starting index of the lower limit is larger than the index of the upper limit. For example, in the above equation, a portion of the summation is zero when k=K because the lower limit is K+1 and the upper limit is K. This applies to all summations in this note.

The last discretization is the vertical flux, which can be derived as

$$\left(\dot{\zeta} \frac{\partial ()}{\partial \zeta} \right)_k = \left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \frac{\partial ()}{\partial p} \right)_k = \left(\frac{\partial \dot{\zeta} \frac{\partial p}{\partial \zeta} ()}{\partial p} - () \frac{\partial \dot{\zeta} \frac{\partial p}{\partial \zeta}}{\partial p} \right)_k$$

$$= \frac{1}{\hat{p}_{k+1} - \hat{p}_k} \left[\widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_{k+1} \frac{()_{k+1} + ()_k}{2} - \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_k \frac{()_k + ()_{k-1}}{2} - ()_k \left\langle \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_{k+1} - \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_k \right\rangle \right] \tag{4.25}$$

$$= \frac{1}{2(\hat{p}_{k+1} - \hat{p}_k)} \left[\widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_{k+1} (()_{k+1} - ()_k) + \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_k (()_k - ()_{k-1}) \right]$$

$$= \frac{1}{2} \left\langle \underline{\underline{ \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_k \frac{()_{k-1} - ()_k}{\hat{p}_k - \hat{p}_{k+1}} }} + \overline{\overline{ \widehat{\left(\dot{\zeta} \frac{\partial p}{\partial \zeta} \right)}_{k+1} \frac{()_k - ()_{k+1}}{\hat{p}_k - \hat{p}_{k+1}} }} \right\rangle$$

where k=1,K with boundary conditions indicated by the double underbar and/or double overbar. From here on, the double underbar in the equation indicates that those terms are zero for the bottom boundary, and the double overbar indicates zero terms for the top boundary condition. For example, in Eq. (4.25) the bottom boundary condition term with the double underbar is zero when k=1, and when k=K the top boundary condition term with the double overbar is zero; for all other k's these two terms are both used. Therefore, the completed set of equations can be discretized as in the following, for all k with boundary conditions at k=1 and k=K:

$$\frac{\partial u_k^*}{\partial t}=-m^2u_k^*\frac{\partial u_k^*}{a\partial\lambda}-m^2v_k^*\frac{\partial u_k^*}{a\partial\varphi}-\frac{1}{2}\left\langle\underline{\underline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_k\frac{u_{k-1}^*-u_k^*}{\hat{p}_k-\hat{p}_{k+1}}}}+\overline{\overline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_{k+1}\frac{u_k^*-u_{k+1}^*}{\hat{p}_k-\hat{p}_{k+1}}}}\right\rangle$$

$$-\frac{R_dT_{vk}}{\hat{p}_k+\hat{p}_{k+1}}\left[\frac{\partial\hat{p}_k+\hat{p}_{k+1}}{a\partial\lambda}\right]-\frac{\partial\Phi_s}{a\partial\lambda} \tag{4.26a}$$

$$-\sum_{i=1}^{k-1}\frac{R_dT_{vi}}{\hat{p}_i+\hat{p}_{i+1}}\left[\frac{\partial\hat{p}_i-\hat{p}_{i+1}}{a\partial\lambda}-\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial\hat{p}_i+\hat{p}_{i+1}}{a\partial\lambda}\right]-\sum_{i=1}^{k}\frac{R_dT_{vi}}{\hat{p}_i+\hat{p}_{i+1}}\left[\frac{\partial\hat{p}_i-\hat{p}_{i+1}}{a\partial\lambda}-\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial\hat{p}_i+\hat{p}_{i+1}}{a\partial\lambda}\right]$$

$$-R_d\sum_{i=1}^{k-1}\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial T_{vi}}{a\partial\lambda}-R_d\sum_{i=1}^{k}\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial T_{vi}}{a\partial\lambda}+f_sv_k^*+F_{u_k}^*$$

$$\frac{\partial v_k^*}{\partial t}=-m^2u_k^*\frac{\partial v_k^*}{a\partial\lambda}-m^2v_k^*\frac{\partial v_k^*}{a\partial\varphi}-\frac{1}{2}\left\langle\underline{\underline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_k\frac{v_{k-1}^*-v_k^*}{\hat{p}_k-\hat{p}_{k+1}}}}+\overline{\overline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_{k+1}\frac{v_k^*-v_{k+1}^*}{\hat{p}_k-\hat{p}_{k+1}}}}\right\rangle$$

$$-\frac{R_dT_{vk}}{\hat{p}_k+\hat{p}_{k+1}}\left[\frac{\partial\hat{p}_k-\hat{p}_{k+1}}{a\partial\varphi}\right]-\frac{\partial\Phi_s}{a\partial\varphi} \tag{4.26b}$$

$$-\sum_{i=1}^{k-1}\frac{R_dT_{vi}}{\hat{p}_i+\hat{p}_{i+1}}\left[\frac{\partial\hat{p}_i-\hat{p}_{i+1}}{a\partial\varphi}-\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial\hat{p}_i+\hat{p}_{i+1}}{a\partial\varphi}\right]-\sum_{i=1}^{k}\frac{R_dT_{vi}}{\hat{p}_i+\hat{p}_{i+1}}\left[\frac{\partial\hat{p}_i-\hat{p}_{i+1}}{a\partial\varphi}-\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial\hat{p}_i+\hat{p}_{i+1}}{a\partial\varphi}\right]$$

$$-R_d\sum_{i=1}^{k-1}\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial T_{vi}}{a\partial\varphi}-R_d\sum_{i=1}^{k}\frac{\hat{p}_i-\hat{p}_{i+1}}{\hat{p}_i+\hat{p}_{i+1}}\frac{\partial T_{vi}}{a\partial\varphi}-f_su_k^*-m^2\frac{s_k^{*2}}{a}\sin\phi+F_{v_k}^*$$

$$\frac{\partial T_{vk}}{\partial t}=-m^2u_k^*\frac{\partial T_{vk}}{a\partial\lambda}-m^2v_k^*\frac{\partial T_{vk}}{a\partial\varphi}-\frac{1}{2}\left\langle\underline{\underline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_k\frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}}}+\overline{\overline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_{k+1}\frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}}}\right\rangle$$

$$+\frac{\kappa T_{vk}}{\hat{p}_k+\hat{p}_{k+1}}m^2\left\langle\vec{V}_k\quad\nabla(\hat{p}_k+\hat{p}_{k+1})-\sum_{i=k}^{K}\left((\hat{p}_i-\hat{p}_{i+1})D_i^*+V_i^*\quad\nabla(\hat{p}_i-\hat{p}_{i+1})\right)\right. \tag{4.26c}$$

$$\left.-\sum_{i=k+1}^{K}\left((\hat{p}_i-\hat{p}_{i+1})D_i^*+V_i^*\quad\nabla(\hat{p}_i-\hat{p}_{i+1})\right)\right\rangle+F_{T_{vk}}$$

$$\frac{\partial\hat{p}_k}{\partial t}=-m^2\sum_{i=k}^{K}\left((\hat{p}_i-\hat{p}_{i+1})\left(\frac{\partial u_i^*}{a\partial\lambda}+\frac{\partial v_i^*}{a\partial\varphi}\right)+u_i^*\frac{\partial(\hat{p}_i-\hat{p}_{i+1})}{a\partial\lambda}+v_i^*\frac{\partial(\hat{p}_i-\hat{p}_{i+1})}{a\partial\varphi}\right)-\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_k \tag{4.26d}$$

$$\frac{\partial q_{i_k}}{\partial t}=-m^2u_k^*\frac{\partial q_{i_k}}{a\partial\lambda}-m^2v_k^*\frac{\partial q_{i_k}}{a\partial\varphi}-\frac{1}{2}\left\langle\underline{\underline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_k\frac{q_{i_{k-1}}-q_{i_k}}{\hat{p}_k-\hat{p}_{k+1}}}}+\overline{\overline{\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_{k+1}\frac{q_{i_k}-q_{i_{k+1}}}{\hat{p}_k-\hat{p}_{k+1}}}}\right\rangle+F_{q_{ik}} \tag{4.26e}$$

under the conditions of

$$\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_{K+1}=\left(\dot{\zeta}\frac{\partial \hat{p}}{\partial\zeta}\right)_1=0 \tag{4.27}$$

$$\frac{\partial \hat{p}_{K+1}}{\partial t} = \nabla \hat{p}_{K+1} = 0$$
$$\frac{\partial \hat{p}_1}{\partial t} = \frac{\partial p_s}{\partial t} \tag{4.28}$$

These equations can be closed as the vertical fluxes for all interfaces are given. The vertical fluxes will be given in a general method described in the next section.

5. Vertical coordinate velocity

The last undetermined variable from the previous sections is hybrid coordinate velocity. It can be determined specifically due to the definition of the hybrid coordinate, but here we start to provide a general solution to it. We have a $\sigma=(p/p_{sfc})$ coordinate in the operational model, σ-p hybrid coordinate in the parallel model, and an isentropic hybrid coordinate as a future goal. Thus, let's consider the hybrid coordinate as a function of surface pressure Ps, pressure P, and virtual temperature Tv, because they are used as prognostic variables in the model as

$$\zeta = F(p, p_{sfc}, T_v) \tag{5.1}$$

then the coordinate velocity can be obtained from the derivative of equation (5.1) with respect to time at levels from 1 to K+1; since the coordinate is not a function of time, the time derivation of Eq. (5.1) should be zero with expansion as in the following:

$$\left(\frac{\partial \hat{F}}{\partial t}\right)_k = \left(\frac{\partial \hat{F}}{\partial p_s}\right)_k \frac{\partial p_s}{\partial t} + \left(\frac{\partial \hat{F}}{\partial p}\right)_k \frac{\partial \hat{p}_k}{\partial t} + \left(\frac{\partial \hat{F}}{\partial T_v}\right)_k \frac{\partial \hat{T}_{vk}}{\partial t} = 0 \tag{5.2}$$

The contribution of local time changes can be separated into vertical flux and non-vertical flux terms, so let's start from Eqs. (4.21) and (4.24), which can be written as

$$\begin{aligned}\frac{\partial \hat{p}_k}{\partial t} &= -m^2 \sum_{i=k}^{K} \left((\hat{p}_i - \hat{p}_{i+1}) \left(\frac{\partial u_i^*}{a\partial\lambda} + \frac{\partial v_i^*}{a\partial\varphi} \right) + u_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\lambda} + v_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\varphi} \right) - \left(\hat{\dot{\zeta}} \frac{\partial p}{\partial \zeta} \right)_k \\ &= \left(\frac{\partial \hat{p}_k}{\partial t} \right)_k^H - \left(\hat{\dot{\zeta}} \frac{\partial p}{\partial \zeta} \right)_k\end{aligned} \tag{5.3a}$$

where the superscript H is for non-vertical-flux term,

$$\left(\frac{\partial \hat{p}_k}{\partial t} \right)_k^H = -m^2 \sum_{i=k}^{K} \left((\hat{p}_i - \hat{p}_{i+1}) \left(\frac{\partial u_i^*}{a\partial\lambda} + \frac{\partial v_i^*}{a\partial\varphi} \right) + u_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\lambda} + v_i^* \frac{\partial(\hat{p}_i - \hat{p}_{i+1})}{a\partial\varphi} \right) \tag{5.3b}$$

and for k=2,K since k=1 and K+1, vertical fluxes are zero. In the same way, from Eq. (4.26c), we define

$$\left(\frac{\partial T_v}{\partial t}\right)_k^H = -m^2 u_k^* \frac{\partial T_{vk}}{a\partial\lambda} - m^2 v_k^* \frac{\partial T_{vk}}{a\partial\varphi} + \frac{\kappa T_{vk}}{\hat{p}_k + \hat{p}_{k+1}} m^2 \Big\langle \vec{V}_k \quad \nabla(\hat{p}_k + \hat{p}_{k+1}) \\ - \sum_{i=k}^{K}\big((\hat{p}_i - \hat{p}_{i+1})D_i^* + V_i^* \quad \nabla(\hat{p}_i - \hat{p}_{i+1})\big) - \sum_{i=k+1}^{K}\big((\hat{p}_i - \hat{p}_{i+1})D_i^* + V_i^* \quad \nabla(\hat{p}_i - \hat{p}_{i+1})\big)\Big\rangle + F_{T_{vk}} \tag{5.4a}$$

which can also be separated into adiabatic and diabatic heating terms as

$$\left(\frac{\partial T_v}{\partial t}\right)_{D_k}^H = -m^2 u_k^* \frac{\partial T_{vk}}{a\partial\lambda} - m^2 v_k^* \frac{\partial T_{vk}}{a\partial\varphi} + \frac{\kappa T_{vk}}{\hat{p}_k + \hat{p}_{k+1}} m^2 \Big\langle \vec{V}_k \quad \nabla(\hat{p}_k + \hat{p}_{k+1}) \\ - \sum_{i=k}^{K}\big((\hat{p}_i - \hat{p}_{i+1})D_i^* + V_i^* \quad \nabla(\hat{p}_i - \hat{p}_{i+1})\big) - \sum_{i=k+1}^{K}\big((\hat{p}_i - \hat{p}_{i+1})D_i^* + V_i^* \quad \nabla(\hat{p}_i - \hat{p}_{i+1})\big)\Big\rangle \tag{5.4b}$$

$$\left(\frac{\partial T_v}{\partial t}\right)_{P_k}^H = F_{T_{vk}} \tag{5.4c}$$

so that

$$\frac{\partial T_{vk}}{\partial t} = \left(\frac{\partial T_{vk}}{\partial t}\right)_k^H - \frac{1}{2}\left\langle \underline{\underline{\left(\dot{\zeta}\widehat{\frac{\partial p}{\partial \zeta}}\right)_k \frac{T_{vk-1} - T_{vk}}{\hat{p}_k - \hat{p}_{k+1}}}} + \overline{\overline{\left(\dot{\zeta}\widehat{\frac{\partial p}{\partial \zeta}}\right)_{k+1} \frac{T_{vk} - T_{vk+1}}{\hat{p}_k - \hat{p}_{k+1}}}} \right\rangle \tag{5.5}$$

for k=1,K with boundary conditions for k=1and k=K as indicated by the double underbar and double overbar, respectively. Thus, the temperature tendency at an interface

$$\frac{\partial \hat{T}_{vk}}{\partial t} = \frac{1}{2}\left(\frac{\partial T_{vk}}{\partial t} + \frac{\partial T_{vk-1}}{\partial t}\right) \tag{5.6}$$

can be expressed as

$$\frac{\partial \hat{T}_{vk}}{\partial t} = \frac{1}{2}\left(\left(\frac{\partial T_v}{\partial t}\right)_k^H + \left(\frac{\partial T_v}{\partial t}\right)_{k-1}^H\right) - \overline{\overline{\frac{1}{4}\frac{T_{vk} - T_{vk+1}}{\hat{p}_k - \hat{p}_{k+1}}\left(\dot{\zeta}\widehat{\frac{\partial p}{\partial \zeta}}\right)_{k+1}}} - \frac{1}{4}\left(\frac{T_{vk-1} - T_{vk}}{\hat{p}_k - \hat{p}_{k+1}} + \frac{T_{vk-1} - T_{vk}}{\hat{p}_{k-1} - \hat{p}_k}\right)\left(\dot{\zeta}\widehat{\frac{\partial p}{\partial \zeta}}\right)_k \\ \underline{\underline{-\frac{1}{4}\frac{T_{vk-2} - T_{vk-1}}{\hat{p}_{k-1} - \hat{p}_k}\left(\dot{\zeta}\widehat{\frac{\partial p}{\partial \zeta}}\right)_{k-1}}} \tag{5.7}$$

for interfaces of k=2,K with the boundary equations for k=2 and K as indicated. Next we put Eqs. (5.3) and (5.7) into Eq. (5.2). With some manipulation, we obtain

$$\overline{\overline{\underline{\underline{\frac{1}{4}\frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k+1}+\left[\frac{1}{4}\left(\frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}+\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k}\right)\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\right]\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_k+\frac{1}{4}\frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k-1}}}}}$$
$$=\left(\frac{\partial \hat{F}}{\partial p_s}\right)_k\frac{\partial p_s}{\partial t}+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\left(\frac{\partial \hat{p}}{\partial t}\right)_k^H+\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\frac{1}{2}\left(\left(\frac{\partial T_v}{\partial t}\right)_k^H+\left(\frac{\partial T_v}{\partial t}\right)_{k-1}^H\right) \tag{5.8}$$

for k=2,K with boundary equations at k=2 and K as indicated. This can be solved by matrix inversion. The method to obtain vertical flux is (1) to compute the surface pressure equation and pressure equation (continuity equation) without vertical flux, then (2) to compute the thermodynamic equation without vertical advection, then (3) to substitute pressure and temperature tendencies without vertical flux into the right-hand-side of Eq. (5.8); then vertical flux can be solved, as mentioned above, by matrix inversion.

In the case of time-splitting for the dynamics and physics computations, Eq. (5.8), which requires diabatic heating as shown in Eq. (5.4), can be rewritten into following form by separating the total vertical flux into adiabatic vertical flux and diabatic vertical flux as well as the heating terms into adiabatic and diabatic terms

$$\overline{\overline{\underline{\underline{\frac{1}{4}\frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left\langle\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_{k+1}}+\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{P_{k+1}}\right\rangle+\left[\frac{1}{4}\left(\frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}+\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k}\right)\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\right]\left\langle\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_k}+\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{P_k}\right\rangle+\frac{1}{4}\frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left\langle\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_{k-1}}+\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{P_{k-1}}\right\rangle}}}}$$
$$=\left(\frac{\partial \hat{F}}{\partial p_s}\right)_k\frac{\partial p_s}{\partial t}+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\left(\frac{\partial \hat{p}}{\partial t}\right)_k^H+\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\frac{1}{2}\left(\left(\frac{\partial T_v}{\partial t}\right)_{D_k}^H+\left(\frac{\partial T_v}{\partial t}\right)_{P_k}^H+\left(\frac{\partial T_v}{\partial t}\right)_{D_{k-1}}^H+\left(\frac{\partial T_v}{\partial t}\right)_{P_{k-1}}^H\right) \tag{5.9}$$

during adiabatic computation in dynamics, the contribution of diabatic heating can be removed as

$$\overline{\overline{\underline{\underline{\frac{1}{4}\frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_{k+1}}+\left[\frac{1}{4}\left(\frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}+\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k}\right)\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\right]\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_k}+\frac{1}{4}\frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k}\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{D_{k-1}}}}}}$$
$$=\left(\frac{\partial \hat{F}}{\partial p_s}\right)_k\frac{\partial p_s}{\partial t}+\left(\frac{\partial \hat{F}}{\partial p}\right)_k\left(\frac{\partial \hat{p}}{\partial t}\right)_k^H+\left(\frac{\partial \hat{F}}{\partial T_v}\right)_k\frac{1}{2}\left(\left(\frac{\partial T_v}{\partial t}\right)_{D_k}^H+\left(\frac{\partial T_v}{\partial t}\right)_{D_{k-1}}^H\right) \tag{5.10}$$

and after physics, we have to re-compute vertical flux with diabatic forcing only, and add the vertical advection to all prognostic equations as the same as in the dynamics

$$\overline{\overline{\frac{1}{4}\frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}\left(\frac{\partial\hat{F}}{\partial T_v}\right)_k\left(\dot{\zeta}\frac{\partial\hat{p}}{\partial\zeta}\right)_{p_{k+1}}}}+\left[\frac{1}{4}\left(\frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}+\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k}\right)\left(\frac{\partial\hat{F}}{\partial T_v}\right)_k+\left(\frac{\partial\hat{F}}{\partial p}\right)_k\right]\left(\dot{\zeta}\frac{\partial\hat{p}}{\partial\zeta}\right)_{p_k}$$

$$\underline{\underline{+\frac{1}{4}\frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k}\left(\frac{\partial\hat{F}}{\partial T_v}\right)_k\left(\dot{\zeta}\frac{\partial\hat{p}}{\partial\zeta}\right)_{p_{k-1}}}} \tag{5.11}$$

$$=\left(\frac{\partial\hat{F}}{\partial T_v}\right)_k\frac{1}{2}\left(\left(\frac{\partial T_v}{\partial t}\right)^H_{p_k}+\left(\frac{\partial T_v}{\partial t}\right)^H_{p_{k-1}}\right)$$

6. Semi-implicit time integration and semi-implicit adjustment

After total tendency is computed by the above descritization, let's consider a semi-implicit time scheme (Robert 1969; Robert et al 1972; Hoskins and Simmons 1975, Simmons et al 1978). It is easier to describe the time scheme with divergence equations, so the momentum tendencies can be converted into divergence and vorticity tendencies before computing the semi-implicit integration. From the equation system, it can be seen that we need not only the reference field of temperature but we also need pressure in order to have linear terms for the semi-implicit scheme. Let's define reference pressure and temperature as p_0 and T_0. Then the equations of divergence, temperature and continuity can be linearized as

$$\frac{\partial D_k^*}{\partial t}=-\frac{R_dT_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\nabla^2(\hat{p}_k+\hat{p}_{k+1})-\sum_{i=1}^{k-1}\frac{R_dT_{0i}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\left[\nabla^2(\hat{p}_i-\hat{p}_{i+1})-\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\nabla^2(\hat{p}_i+\hat{p}_{i+1})\right]$$

$$-\sum_{i=1}^{k}\frac{R_dT_{0i}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\left[\nabla^2(\hat{p}_i-\hat{p}_{i+1})-\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\nabla^2(\hat{p}_i+\hat{p}_{i+1})\right] \tag{6.1a}$$

$$-R_d\sum_{i=1}^{k-1}\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\nabla^2(T_v)_i-R_d\sum_{i=1}^{k}\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}\nabla^2(T_v)_i+X'$$

$$\frac{\partial T_{vk}}{\partial t}=-\frac{(R_d/C_{pd})T_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\left(\sum_{i=k}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})+\sum_{i=k+1}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})\right)D_i^*+Y' \tag{6.1b}$$

$$\frac{\partial\hat{p}_k}{\partial t}=-\sum_{i=k}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})D_i^*+Z' \tag{6.1c}$$

where X', Y' and Z' are all nonlinear forcings, the total tendency minus the linear terms. When linearized, the computation can then be done in spectral space, thus the Laplacian operator can be replaced by

$$\nabla^2[T]=-\frac{n(n+1)}{a^2}[T] \tag{6.2}$$

where n is the wave number and [T] is the spectral coefficient for T. For simplicity, [ ] is removed from now on in this section, but all variables are in coefficient form for the spectral space computation. Thus, Eq. (6.1) can be transformed into spectral coefficients and computed in spectral space in a simplified form with some vectors as

$$\frac{\partial D_k^*}{\partial t} = \mathrm{H}_{ki} p_i + \mathrm{A}_{ki} T_{v_i} + X_k' \qquad (6.3a)$$

$$\frac{\partial T_{vk}}{\partial t} = -\mathrm{B}_{ki} D_i^* + Y_k' \qquad (6.3b)$$

$$\frac{\partial \hat{p}_k}{\partial t} = -S_{ki} D_i^* + Z_k' \qquad (6.3c)$$

where A, B, H and S are vectors in index i for a given k as

$$\mathrm{H}_{ki} p_i = \frac{n(n+1)R_d}{a^2} \left\langle \frac{T_{0k}}{\hat{p}_{0k} + \hat{p}_{0k+1}} (\hat{p}_k + \hat{p}_{k+1}) + \sum_{i=1}^{k-1} \frac{T_{0i}}{\hat{p}_{0i} + \hat{p}_{0i+1}} \left[(\hat{p}_i - \hat{p}_{i+1}) - \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}} (\hat{p}_i + \hat{p}_{i+1}) \right] \right.$$
$$\left. + \sum_{i=1}^{k} \frac{T_{0i}}{\hat{p}_{0i} + \hat{p}_{0i+1}} \left[(\hat{p}_i - \hat{p}_{i+1}) - \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}} (\hat{p}_i + \hat{p}_{i+1}) \right] \right\rangle \qquad (6.4a)$$

$$\mathrm{A}_{ki} T_{vi} = \frac{n(n+1)R_d}{a^2} \left\langle \sum_{i=1}^{k-1} \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}} (T_v)_i + \sum_{i=1}^{k} \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}} (T_v)_i \right\rangle \qquad (6.4b)$$

$$\mathrm{B}_{ki} D_i^* = \frac{(R_d / C_{pd}) T_{0k}}{\hat{p}_{0k} + \hat{p}_{0k+1}} \left(\sum_{i=k}^{K} (\hat{p}_{0i} - \hat{p}_{0i+1}) + \sum_{i=k+1}^{K} (\hat{p}_{0i} - \hat{p}_{0i+1}) \right) D_i^* \qquad (6.4c)$$

$$S_{ki} D_i^* = \sum_{i=k}^{K} (\hat{p}_{0i} - \hat{p}_{0i+1}) D_i^* \qquad (6.4d)$$

or they can matrix in all indexes i and k if all prognostic variables are represented as vectors for all k (see Appendix A for an example with 4 layers) which can be easily expanded to any number of layers.

Next, let's turn the linear terms into a semi-implicit scheme by

$$\frac{()^{n+1} + ()^{n-1}}{2} = \bar{()}$$
$$\frac{\partial ()}{\partial t} = \frac{()^{n+1} - ()^{n-1}}{2\Delta t} = \frac{1}{\Delta t} \left(\bar{()} - ()^{n-1} \right) \qquad (6.5)$$

thus Eq. (6.3) becomes

$$\overline{D}_k^* = D_k^{*n-1} + \Delta t \left(\mathrm{H}_{ki} \overline{p}_i + \mathrm{A}_{ki} \overline{T}_{v_i} + X_k' \right)$$
$$= D_k^{*n-1} + \Delta t \left(\mathrm{H}_{ki} \overline{p}_i + \mathrm{A}_{ki} \overline{T}_{v_i} + X_k - \mathrm{H}_{ki} p_i^n - \mathrm{A}_{ki} T_{v_i}^n \right) \qquad (6.6a)$$

$$
\begin{aligned}
\overline{T}_{vk} &= T_{vk}^{\ n-1} - \Delta t\left(\mathrm{B}_{ki}\overline{D}_i^* - Y_k^{'}\right) \\
&= T_{vk}^{\ n-1} - \Delta t\left(\mathrm{B}_{ki}\overline{D}_i^* - Y_k - \mathrm{B}_{ki}D_i^{*n}\right)
\end{aligned} \tag{6.6b}
$$

$$
\begin{aligned}
\overline{p}_k &= p_k^{\ n-1} - \Delta t\left(S_{ki}\overline{D}_i^* - Z_k^{'}\right) \\
&= p_k^{\ n-1} - \Delta t\left(S_{ki}\overline{D}_i^* - Z_k - S_{ki}D_i^{*n}\right)
\end{aligned} \tag{6.6c}
$$

where X, Y, Z are total tendencies and k=1,K. To solve this, we put Eqs. (6.6b) and (6.6c) into Eq. (6.6a), after some manipulations, we get

$$
\begin{aligned}
\overline{D}_k^* &= \left(1 + \Delta t^2\left\langle \mathrm{H}_{ki}S_{ij} + \mathrm{A}_{ki}\mathrm{B}_{ij}\right\rangle\right)^{-1} \\
&\left\{D_k^{*n-1} + \Delta t\left(X_k + \mathrm{H}_{ki}\left\langle p_i^{n-1} - p_i^{n} + \Delta t\left(Z_i + S_{ij}D_j^{*n}\right)\right\rangle + \mathrm{A}_{ki}\left\langle T_{vi}^{\ n-1} - T_{vi}^{\ n} + \Delta t\left(Y_i + \mathrm{B}_{ij}D_j^{*n}\right)\right\rangle\right)\right\}
\end{aligned} \tag{6.7}
$$

then we put the resolved divergence from Eq. (6.7) back into Eqs. (6.6b) and (6.6c), and we can solve T and p. It can be simplified and made computationally more efficient as

$$
\tilde{T}_{vk} = T_{vk}^{\ n-1} + \Delta t\left(Y_k + \mathrm{B}_{ki}D_i^{*n}\right) \tag{6.8a}
$$

$$
\tilde{p}_k = p_k^{\ n-1} + \Delta t\left(Z_k + S_{ki}D_i^{*n}\right) \tag{6.8b}
$$

then we solve the mean values

$$
\overline{D}_k^* = \left(1 + \Delta t^2\left\langle \mathrm{H}_{ki}S_{ij} + \mathrm{A}_{ki}\mathrm{B}_{ij}\right\rangle\right)^{-1}\left\{D_k^{*n-1} + \Delta t\left(X_k + \mathrm{H}_{ki}\left\langle \tilde{p}_i - p_i^{n}\right\rangle + \mathrm{A}_{ki}\left\langle \tilde{T}_{vi} - T_{vi}^{\ n}\right\rangle\right)\right\} \tag{6.9a}
$$

$$
\overline{T}_{vk} = \tilde{T}_{vk} - \Delta t\mathrm{B}_{ki}\overline{D}_i^* \tag{6.9b}
$$

$$
\overline{p}_k = \tilde{p}_k - \Delta t S_{ki}\overline{D}_i^* \tag{6.9c}
$$

Next, let's consider a semi-implicit adjustment after physics. We can start from the same equation set as in Eq. (6.2), but with an adjustment between the values after dynamics at n+1 and adjusted values with physics forcing at n+1. The notation for the semi-implicit adjustment are as follows:

$$
\begin{aligned}
&\frac{()^{n+1} + ()^{(n+1)_d}}{2} = \overline{()} \\
&\frac{\partial()}{\partial t} = \frac{()^{n+1} - ()^{(n+1)_d}}{2\Delta t} = \frac{\Delta()}{2\Delta t} \\
&\Delta() = ()^{n+1} - ()^{(n+1)_d}
\end{aligned} \tag{6.10}
$$

where $(n+1)_d$ denotes the value after dynamics, n+1 denotes the value after physics and adjustment. The nonlinear terms in Eq. (6.3) are

$$X' = \left(\partial D^* / \partial t\right)_{physics} - \mathrm{H}p^{(n+1)_d} - \mathrm{A}T^{(n+1)_d} = X - \mathrm{H}p^{(n+1)_d} - \mathrm{A}T^{(n+1)_d} \tag{6.11a}$$

$$Y' = \left(\partial T / \partial t\right)_{physics} + \mathrm{B}D^{(n+1)_d} = Y + \mathrm{B}D^{(n+1)_d} \tag{6.11b}$$

$$Z' = \left(\partial p / \partial t\right)_{physics} + SD^{(n+1)_d} = Z + SD^{(n+1)_d} \tag{6.11c}$$

and when Eqs. (6.10) and (6.11) are put into Eq. (6.3), we obtain

$$\Delta D_k^* = \Delta t\left(\mathrm{H}_{ki}\Delta p_i + \mathrm{A}_{ki}\Delta T_{v_i}\right) + 2\Delta t X_k = \Delta t\left(\mathrm{H}_{ki}\Delta p_i + \mathrm{A}_{ki}\Delta T_{v_i}\right) + \left(\Delta D_k^*\right)_{physics} \tag{6.12a}$$

$$\Delta T_{vk} = -\Delta t \mathrm{B}_{ki}\Delta D_i^* + 2\Delta t Y_k = -\Delta t \mathrm{B}_{ki}\Delta D_i^* + \left(\Delta T_k\right)_{physics} \tag{6.12b}$$

$$\Delta p_k = -\Delta t S_{ki}\Delta D_i^* + 2\Delta t Z_k = -\Delta t S_{ki}\Delta D_i^* + \left(\Delta p_k\right)_{physics} \tag{6.12c}$$

where the last terms represented are changed from physics computation, and the others are the differences between the adjusted and after-dynamics cases. Again, after using the same reduction by substituting Eqs. (6.12b) and (6.12c) into Eq. (6.12a), we obtain

$$\Delta D_j^* = \left(1 + \Delta t^2\left\langle \mathrm{H}_{ki}S_{ij} + \mathrm{A}_{ki}\mathrm{B}_{ij}\right\rangle\right)^{-1}\left[\Delta t\left(\mathrm{H}_{ki}\left(\Delta p_i\right)_{physics} + \mathrm{A}_{ki}\left(\Delta T_i\right)_{physics}\right) + \left(\Delta D_k^*\right)_{physics}\right] \tag{6.13}$$

then when we substitute Eq. (6.13) into Eqs. (6.12b) and (6.12c), we can obtain T and p.

In practicality, we can have only one routine to do semi-implicit adjustments for both dynamics and physics. The routine can be designed in such a way to pass in the total tendency and all three-time-level prognostic variables for divergence, temperature and pressure. After dynamics, the total tendencies and prognostic variables for time level n-1 and n are passed into the routine, and the third time-level prognostic values at $(n+1)_d$ are returned. After physics, the changes due to physics and the $(n+1)_d$ prognostic variables after dynamics are passed into the n-1 and n time-level variables, and n+1 time level is returned. However, physics adjustments can be done with less computation by not using the same routine as dynamics adjustment, as seen in a comparison of Eq. (6.13) to Eq. (6.7).

7. Specific hybrid coordinates

Until the previous section, the finite difference equation sets, Eqs. (4.26), (5.8), (6.7) and (6.13), are for generalized hybrid coordinates, which can be used for sigma, sigma-pressure, sigma-theta, and sigma-theta-pressure, etc. When a specific coordinate is used, these finite difference equations can be used with some modification. the following is a specific one that generally covers sigma, pressure and/or isentropic as

$$\hat{p}_k = \hat{A}_k + \hat{B}_k p_s + \hat{C}_k\left(\hat{T}_{vk} / \hat{T}_{0k}\right)^{C_p / R_d} \tag{7.1}$$

where A, B and C are specified and constant during integration with the following known boundary conditions for pressure at the top of the atmosphere and at the surface, as

$$\hat{A}_{K+1} = \hat{B}_{K+1} = \hat{C}_{K+1} = 0$$
$$\hat{A}_1 = \hat{C}_1 = 0 \qquad (7.2)$$
$$\hat{B}_1 = 1$$

The pressure and gradient of pressure for k=2,K at the interfaces can be written as

$$\hat{p}_k = \hat{A}_k + \hat{B}_k p_s + \hat{C}_k \left(\frac{T_{vk-1} + T_{vk}}{T_{0k-1} + T_{0k}} \right)^{C_p/R_d} \qquad (7.3a)$$

$$\frac{\partial \hat{p}_k}{a\partial s} = \hat{B}_k \frac{\partial p_s}{a\partial s} + \frac{\hat{C}_k}{T_{vk-1} + T_{vk}} \frac{C_p}{R_d} \left(\frac{T_{vk-1} + T_{vk}}{T_{0k-1} + T_{0k}} \right)^{C_p/R_d} \left(\frac{\partial T_{vk-1}}{a\partial s} + \frac{\partial T_{vk}}{a\partial s} \right) \qquad (7.3b)$$

where s can be either latitude and longitude, and all of which are zero when k=K+1. For k=1, they are

$$\hat{p}_1 = p_s$$
$$\frac{\partial \hat{p}_1}{a\partial s} = \frac{\partial p_s}{a\partial s} \qquad (7.4)$$

because A=C=0 and B=1 at level k=1. After pressure and its derivatives are computed for all interfaces by Eq. (7.3), Eq. (4.26) can be used, but the continuity equation in (4.26d) will be used to solve Ps and vertical flux later. The surface pressure equation, as k=1 in Eq. (4.26d), and is as follows:

$$\frac{\partial p_s}{\partial t} = -m^2 \sum_{i=1}^{K} \left((\hat{p}_i - \hat{p}_{i+1}) \left(\frac{\partial u_i^*}{a\partial \lambda} + \frac{\partial v_i^*}{a\partial \varphi} \right) + u_i^* \frac{\partial (\hat{p}_i - \hat{p}_{i+1})}{a\partial \lambda} + v_i^* \frac{\partial (\hat{p}_i - \hat{p}_{i+1})}{a\partial \varphi} \right) \qquad (7.5)$$

and the vertical flux can be solved starting from the following

$$\frac{\partial \hat{p}_k}{\partial t} = \hat{B}_k \frac{\partial p_s}{\partial t} + \frac{\hat{C}_k}{T_{vk-1} + T_{vk}} \frac{C_p}{R_d} \left(\frac{T_{vk-1} + T_{vk}}{T_{0k-1} + T_{0k}} \right)^{C_p/R_d} \left(\frac{\partial T_{vk-1}}{\partial t} + \frac{\partial T_{vk}}{\partial t} \right) \qquad (7.6a)$$

for k=2,K with boundary conditions

$$\frac{\partial \hat{p}_1}{\partial t} = \hat{B}_1 \frac{\partial p_s}{\partial t} \qquad (7.6b)$$

$$\frac{\partial \hat{p}_{K+1}}{\partial t} = 0 \qquad (7.6c)$$

then follow the process in Section 5, separating the vertical flux and non-vertical flux forcings by substituting prognostic equations of surface pressure, pressure, and temperature, we have

$$
\begin{aligned}
\left(\frac{\partial \hat{p}}{\partial t}\right)^{H}_{k} - \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k} = {} & \hat{B}_k \frac{\partial p_s}{\partial t} + \frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \\
& \left\{ \left(\frac{\partial T_v}{\partial t}\right)^{H}_{k} - \frac{1}{2}\left\langle \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k} \frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}} + \overline{\overline{\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k+1} \frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}}}} \right\rangle + \right. \\
& \left. \left(\frac{\partial T_v}{\partial t}\right)^{H}_{k-1} - \frac{1}{2}\left\langle \underline{\underline{\left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k-1} \frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k}}} + \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k} \frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k} \right\rangle \right\}
\end{aligned}
\tag{7.7}
$$

for k=2,K with boundary conditions at k=2 and K indicated by the double underbar and double overbar, which are zero at these respective boundary conditions. After a rearrangement, we have

$$
\begin{aligned}
& \overline{\overline{\frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k+1}}} \\
& + \left\{ \frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \left[\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k} + \frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}\right] - 1 \right\} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k} \\
& + \underline{\underline{\frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{k-1}}} \\
& = \hat{B}_k \frac{\partial p_s}{\partial t} - \left(\frac{\partial \hat{p}}{\partial t}\right)^{H}_{k} + \frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \left[\left(\frac{\partial T_v}{\partial t}\right)^{H}_{k-1} + \left(\frac{\partial T_v}{\partial t}\right)^{H}_{k}\right]
\end{aligned}
\tag{7.8a}
$$

for dynamics while the heating is adiabatic, and the following for physics while heating is diabatic only and no changes made in terms of pressure

$$
\begin{aligned}
& \overline{\overline{\frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \frac{T_{vk}-T_{vk+1}}{\hat{p}_k-\hat{p}_{k+1}} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{Pk+1}}} \\
& + \left\{ \frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \left[\frac{T_{vk-1}-T_{vk}}{\hat{p}_{k-1}-\hat{p}_k} + \frac{T_{vk-1}-T_{vk}}{\hat{p}_k-\hat{p}_{k+1}}\right] - 1 \right\} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{Pk} \\
& + \underline{\underline{\frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{2R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \frac{T_{vk-2}-T_{vk-1}}{\hat{p}_{k-1}-\hat{p}_k} \left(\widehat{\dot{\zeta}\frac{\partial p}{\partial \zeta}}\right)_{Pk-1}}} \\
& = \frac{\hat{C}_k}{T_{vk-1}+T_{vk}} \frac{C_p}{R_d}\left(\frac{T_{vk-1}+T_{vk}}{T_{0k-1}+T_{0k}}\right)^{C_p/R_d} \left[\left(\frac{\partial T_v}{\partial t}\right)^{H}_{Pk-1} + \left(\frac{\partial T_v}{\partial t}\right)^{H}_{Pk}\right]
\end{aligned}
\tag{7.8b}
$$

where k=2,K with boundary computations at k=2 and K indicated by the double underbar and double overbar. This can be solved by matrix inversion.

For a semi-implicit scheme, we start from the linearized horizontal Laplacian of pressure for Eq. (7.3) as

$$
\begin{aligned}
\nabla^2 \hat{p}_k &= \hat{B}_k \nabla^2 p_s + \frac{\hat{C}_k}{\kappa_d \left(T_{0k-1} + T_{0k}\right)} \left(\nabla^2 T_{vk-1} + \nabla^2 T_{vk}\right) \\
&= \hat{B}_k \nabla^2 p_s + \underline{\underline{\overline{\overline{\hat{C}_{0k}\left(\nabla^2 T_{vk-1} + \nabla^2 T_{vk}\right)}}}}
\end{aligned}
\tag{7.9}
$$

for k=1,K+1 with k=1 and K+1 as boundary conditions. From Eq. (6.1), we know we need the following

$$
\nabla^2\left(\hat{p}_k \pm \hat{p}_{k+1}\right) = \left(\hat{B}_k \pm \hat{B}_{k+1}\right)\nabla^2 p_s + \underline{\underline{\hat{C}_{0k}\nabla^2 T_{vk-1}}} + \left(\underline{\underline{\hat{C}_{0k}}} \pm \overline{\overline{\hat{C}_{0k+1}}}\right)\nabla^2 T_{vk} \pm \overline{\overline{\hat{C}_{0k+1}\nabla^2 T_{vk+1}}} \tag{7.10}
$$

for k=1,K with boundary conditions at k=1 and K as indicated. Then we put Eq. (7.10) into Eq. (6.1a) and obtain

$$
\begin{aligned}
\frac{\partial D_k^*}{\partial t} = &-\left[\frac{R_d T_{0k}}{\hat{p}_{0k} + \hat{p}_{0k+1}}\left(\hat{B}_k + \hat{B}_{k+1}\right) + \sum_{i=1}^{k-1} \frac{2R_d T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\left(\hat{B}_i \hat{p}_{0i+1} - \hat{B}_{i+1}\hat{p}_{0i}\right) + \sum_{i=1}^{k} \frac{2R_d T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\left(\hat{B}_i \hat{p}_{0i+1} - \hat{B}_{i+1}\hat{p}_{0i}\right)\right]\nabla^2 p_s \\
&- \frac{R_d T_{0k}}{\hat{p}_{0k} + \hat{p}_{0k+1}}\left[\underline{\underline{\hat{C}_{0k}\nabla^2 T_{vk-1}}} + \left(\underline{\underline{\hat{C}_{0k}}} + \overline{\overline{\hat{C}_{0k+1}}}\right)\nabla^2 T_{vk} + \overline{\overline{\hat{C}_{0k+1}\nabla^2 T_{vk+1}}}\right] \\
&- \sum_{i=1}^{k-1} \frac{2R_d T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\left[\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}\nabla^2 T_{vi-1}}} + \left(\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}}} - \hat{C}_{0i+1}\hat{p}_{0i}\right)\nabla^2 T_{vi} - \hat{C}_{0i+1}\hat{p}_{0i}\nabla^2 T_{vi+1}\right] \\
&- \sum_{i=1}^{k} \frac{2R_d T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\left[\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}\nabla^2 T_{vi-1}}} + \left(\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}}} - \overline{\overline{\hat{C}_{0i+1}\hat{p}_{0i}}}\right)\nabla^2 T_{vi} - \overline{\overline{\hat{C}_{0i+1}\hat{p}_{0i}\nabla^2 T_{vi+1}}}\right] \\
&- R_d \sum_{i=1}^{k-1} \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}}\nabla^2\left(T_v\right)_i - R_d \sum_{i=1}^{k} \frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}}\nabla^2\left(T_v\right)_i + X'
\end{aligned}
\tag{7.11}
$$

for k=1,K with boundary conditions at k=1 and K as indicated. Thus, Eq. (7.11) will modify the matrix H in Eq. (6.3a) into a vector, and will modify the matrix A in Eq. (6.3a) into a different matrix, Eq. (6.3b) will not be changed, and matrix S in Eq. (6.3c) will become a vector. Finally, we have

$$
\frac{\partial D_k^*}{\partial t} = H_{k1}^+ p_s + A_{ki}^+ T_{vi} + X_k{}' \tag{7.12a}
$$

$$
\frac{\partial T_{vk}}{\partial t} = -B_{ki} D_i^* + Y_k{}' \tag{7.12b}
$$

$$
\frac{\partial p_s}{\partial t} = -S_{1i} D_i^* + Z_1{}' \tag{7.12c}
$$

for k=1,K. The matrixes and vectors will be given in detail in Appendix B, with an example using 4 layers, which can be expended to any number of layers. Then the same process described in Section 6 will be used to solve the equation in a semi-implicit method. After applying Eq. (6.7) to make a semi-implicit integration, Eq. (7.12) will be written as

$$\overline{D}_k^* = D_k^{*n-1} + \Delta t\left(\mathrm{H}_{k1}^+\overline{p}_s + \mathrm{A}_{ki}^+\overline{T}_{v_i} + X_k - \mathrm{H}_{k1}^+p_s^n - \mathrm{A}_{ki}^+T_{v_i}^n\right) \tag{7.13a}$$

$$\overline{T}_{vk} = T_{vk}^{\ n-1} - \Delta t\left(\mathrm{B}_{ki}\overline{D}_i^* - Y_k - \mathrm{B}_{ki}D_i^{*n}\right) \tag{7.13b}$$

$$\overline{p}_s = p_s^{\ n-1} - \Delta t\left(S_{1i}\overline{D}_i^* - Z_1 - S_{1i}D_i^{*n}\right) \tag{7.13c}$$

and the reduction of these algebra equations can be simplified to solve

$$\overline{D}_k^* = \left(1 + \Delta t^2\left\langle \mathrm{H}_{k1}^+S_{1j} + \mathrm{A}_{ki}^+\mathrm{B}_{ij}\right\rangle\right)^{-1} \left\{D_k^{*n-1} + \Delta t\left(X_k + \mathrm{H}_{k1}^+\left\langle p_s^{\ n-1} - p_s^{\ n} + \Delta t\left(Z_1 + S_{1j}D_j^{*n}\right)\right\rangle + \mathrm{A}_{ki}^+\left\langle T_{vi}^{\ n-1} - T_{vi}^{\ n} + \Delta t\left(Y_i + \mathrm{B}_{ij}D_j^{*n}\right)\right\rangle\right)\right\} \tag{7.14}$$

first, then Eqs. (7.13b) and (7.13c) are solved by the solution of Eq. (7.14). It is the same for the semi-implicit scheme after physics computation, from Eqs. (6.12) and (6.13), as

$$\Delta D_k^* = \Delta t\left(\mathrm{H}_{k1}^+\Delta p_s + \mathrm{A}_{ki}^+\Delta T_{v_i}\right) + 2\Delta t X_k = \Delta t\left(\mathrm{H}_{k1}^+\Delta p_s + \mathrm{A}_{ki}^+\Delta T_{v_i}\right) + \left(\Delta D_k^*\right)_{physics} \tag{7.15a}$$

$$\Delta T_{vk} = -\Delta t \mathrm{B}_{ki}\Delta D_i^* + 2\Delta t Y_k = -\Delta t\mathrm{B}_{ki}\Delta D_i^* + \left(\Delta T_k\right)_{physics} \tag{7.15b}$$

$$\Delta p_s = -\Delta t S_{1i}\Delta D_i^* + 2\Delta t Z_s = -\Delta t S_{1i}\Delta D_i^* + \left(\Delta p_s\right)_{physics} \tag{7.15c}$$

where the last terms represented are changed from the physics computation and other terms are the differences between the adjusted and after-dynamics cases. Again, after the same reduction by substituting Eqs. (6.12b) and (6.12c) into Eq. (6.12a), we obtain

$$\Delta D_k^* = \left(1 + \Delta t^2\left\langle \mathrm{H}_{k1}^+S_{1j} + \mathrm{A}_{ki}^+\mathrm{B}_{ij}\right\rangle\right)^{-1}\left[\Delta t\left(\mathrm{H}_{ki}^+\left(\Delta p_i\right)_{physics} + \mathrm{A}_{ki}^+\left(\Delta T_i\right)_{physics}\right) + \left(\Delta D_k^*\right)_{physics}\right] \tag{7.16}$$

then we substitute Eq. (7.16) into Eqs. (7.15b) and (7.15c) to obtain T and p.

8. Conclusion

A detailed description of the equation set for a finite difference scheme in the vertical was provided here to show easy to do model programming. This set of finite difference equations conserves mass, energy and angular momentum. The simple relations to satisfy conservation were chosen so that the pressure at a layer is averaged by the neighboring pressures at interfaces, one from the interface immediately above and another from the interface immediately below. All other equations are derived by using this relation.

The generalized hybrid vertical coordinates in the finite difference scheme are derived, including the processes to solve vertical fluxes at all levels (or interfaces), using semi-implicit integration after the dynamical and physical computations for time splitting. This generalized finite difference equation set, including semi-implicit matrixes, can be used for implememting coordinates used in the RUC (Benjamin et al 2004) or University of Wisconsin model (Johnson et al 2003).

A generalized hybrid coordinate from a linear combination of surface pressure and isentropic quantity in terms of virtual temperature is given. Due to the definition of the coordinate, the pressure equation as continuity equation for all interfaces is reduced to a surface pressure equation, thus, the finite difference form of the equation set derived for generalized hybrid coordinates has to be modified in order to take the advantage of the computational saving. The matrixes for pressure are reduced as vectors for surface pressure.

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Appendix A
Matrixes for semi-implicit scheme

From Eqs. (6.1) and (6.3), we have following relation for all matrixes as

$$
\begin{aligned}
\mathrm{H}_{ki}p_i &= \frac{n(n+1)R_d}{a^2}\left\langle \frac{T_{0k}(\hat{p}_k+\hat{p}_{k+1})}{\hat{p}_{0k}+\hat{p}_{0k+1}} + \sum_{i=1}^{k-1}\frac{2T_{0i}(\hat{p}_{0i+1}\hat{p}_i-\hat{p}_{0i}\hat{p}_{i+1})}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2} + \sum_{i=1}^{k}\frac{2T_{0i}(\hat{p}_{0i+1}\hat{p}_i-\hat{p}_{0i}\hat{p}_{i+1})}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2}\right\rangle \\
&= \frac{n(n+1)R_d}{a^2}\left\langle \sum_{i=1}^{k-1}\left(\frac{4T_{0i}(\hat{p}_{0i+1}\hat{p}_i-\hat{p}_{0i}\hat{p}_{i+1})}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2}\right) + \frac{2T_{0k}(\hat{p}_{0k+1}\hat{p}_k-\hat{p}_{0k}\hat{p}_{k+1})}{(\hat{p}_{0k}+\hat{p}_{0k+1})^2} + \frac{T_{0k}(\hat{p}_k+\hat{p}_{k+1})}{\hat{p}_{0k}+\hat{p}_{0k+1}}\right\rangle \\
&= \frac{n(n+1)R_d}{a^2}\left\langle \sum_{i=1}^{k-1}\left(\frac{4T_{0i}(\hat{p}_{0i+1}\hat{p}_i-\hat{p}_{0i}\hat{p}_{i+1})}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2}\right) + \frac{T_{0k}(\hat{p}_{0k}+3\hat{p}_{0k+1})}{(\hat{p}_{0k}+\hat{p}_{0k+1})^2}\hat{p}_k - \frac{T_{0k}(\hat{p}_{0k}-\hat{p}_{0k+1})}{(\hat{p}_{0k}+\hat{p}_{0k+1})^2}\hat{p}_{k+1}\right\rangle
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
A_{ki}T_{vi} &= \frac{n(n+1)R_d}{a^2}\left\langle \sum_{i=1}^{k-1}\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}(T_v)_i + \sum_{i=1}^{k}\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}(T_v)_i\right\rangle \\
&= \frac{n(n+1)R_d}{a^2}\left\langle \sum_{i=1}^{k-1}2\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}(T_v)_i + \frac{\hat{p}_{0k}-\hat{p}_{0k+1}}{\hat{p}_{0k}+\hat{p}_{0k+1}}(T_v)_k\right\rangle
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
B_{ki}D_i^* &= \frac{\kappa_d T_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\left(\sum_{i=k}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1}) + \sum_{i=k+1}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})\right)D_i^* \\
&= \frac{\kappa_d T_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\left((\hat{p}_{0k}-\hat{p}_{0k+1}) + \sum_{i=k+1}^{K}2(\hat{p}_{0i}-\hat{p}_{0i+1})\right)D_i^*
\end{aligned}
\tag{A.3}
$$

$$
S_{ki}D_i^* = \sum_{i=k}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})D_i^* \tag{A.4}
$$

for k=1,K, where Eq. (A.1) is not straightforward when writing all k because the first term in summation has pressure related to a future index, i+1, which has to expand. We can separate all terms into sub-matrixes by manipulating all terms, then sum individual elements after expanding in 4x4 matrixes (as an example).

So, let's re-write Eq. (A.1) into

$$
\mathrm{H}_{ki}p_i = \frac{n(n+1)R_d}{a^2}\left\langle \mathrm{H}^1_{ki}p_i + \mathrm{H}^2_{ki}p_i + \mathrm{H}^3_{ki}p_i\right\rangle \tag{A.5}
$$

where

$$
\mathrm{H}^1_{ki}p_i = \sum_{i=1}^{k-1}\left(\frac{4T_{0i}\hat{p}_{0i+1}}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2}\hat{p}_i\right) \tag{A.6}
$$

$$
\mathrm{H}^2_{ki}p_i = -\sum_{i=1}^{k-1}\left(\frac{4T_{0i}\hat{p}_{0i}}{(\hat{p}_{0i}+\hat{p}_{0i+1})^2}\hat{p}_{i+1}\right) \tag{A.7}
$$

$$\mathrm{H}_{ki}^{3}p_i = \frac{T_{0k}\left(\hat{p}_{0k} + 3\hat{p}_{0k+1}\right)}{\left(\hat{p}_{0k} + \hat{p}_{0k+1}\right)^2}\hat{p}_k - \frac{T_{0k}\left(\hat{p}_{0k} - \hat{p}_{0k+1}\right)}{\left(\hat{p}_{0k} + \hat{p}_{0k+1}\right)^2}\hat{p}_{k+1} \tag{A.8}$$

so, for 4x4 as an example, we have

$$\mathrm{H}^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 & 0 \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{4T_{02}\hat{p}_{03}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 & 0 \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{4T_{02}\hat{p}_{03}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{4T_{03}\hat{p}_{04}}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & 0 \end{bmatrix} \tag{A.9}$$

$$\mathrm{H}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 \\ 0 & -\frac{4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & -\frac{4T_{02}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 \\ 0 & -\frac{4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & -\frac{4T_{02}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & -\frac{4T_{03}\hat{p}_{03}}{\left(\hat{p}_{04}+\hat{p}_{05}\right)^2} \end{bmatrix} \tag{A.10}$$

$$\mathrm{H}^3 = \begin{bmatrix} \frac{T_{01}\left(\hat{p}_{01}+3\hat{p}_{02}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & -\frac{T_{01}\left(\hat{p}_{01}-\hat{p}_{02}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 \\ 0 & \frac{T_{02}\left(\hat{p}_{02}+3\hat{p}_{03}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & -\frac{T_{02}\left(\hat{p}_{02}-\hat{p}_{03}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 \\ 0 & 0 & \frac{T_{03}\left(\hat{p}_{03}+3\hat{p}_{04}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & -\frac{T_{03}\left(\hat{p}_{03}-\hat{p}_{04}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} \\ 0 & 0 & 0 & \frac{T_{04}\left(\hat{p}_{04}+3\hat{p}_{05}\right)}{\left(\hat{p}_{04}+\hat{p}_{05}\right)^2} \end{bmatrix} \tag{A.11}$$

Then, we can sum them together by each element as

$$\mathrm{H} = \frac{n(n+1)}{a^2}R_d \begin{bmatrix} \frac{T_{01}\left(\hat{p}_{01}+3\hat{p}_{02}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-T_{01}\left(\hat{p}_{01}-\hat{p}_{02}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2}+\frac{T_{02}\left(\hat{p}_{02}+3\hat{p}_{03}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{-T_{02}\left(\hat{p}_{02}-\hat{p}_{03}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2}+\frac{4T_{02}\hat{p}_{03}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{-4T_{02}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2}+\frac{T_{03}\left(\hat{p}_{03}+3\hat{p}_{04}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & \frac{-T_{03}\left(\hat{p}_{03}-\hat{p}_{04}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} \\ \frac{4T_{01}\hat{p}_{02}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-4T_{01}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2}+\frac{4T_{02}\hat{p}_{03}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{-4T_{02}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2}+\frac{4T_{03}\hat{p}_{04}}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & \frac{-4T_{03}\hat{p}_{03}}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2}+\frac{T_{04}\left(\hat{p}_{04}+3\hat{p}_{05}\right)}{\left(\hat{p}_{04}+\hat{p}_{05}\right)^2} \end{bmatrix} \tag{A.12}$$

Other matrixes A, B and S can be easily figured out from Eqs (A.2), (A.3) and (A.4) as in the following

$$A = \frac{n(n+1)}{a^2} R_d \begin{bmatrix} \frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 0 & 0 & 0 \\ 2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & \frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & 0 & 0 \\ 2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 2\frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & \frac{\hat{p}_{03}-\hat{p}_{04}}{\hat{p}_{03}+\hat{p}_{04}} & 0 \\ 2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 2\frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & 2\frac{\hat{p}_{03}-\hat{p}_{04}}{\hat{p}_{03}+\hat{p}_{04}} & \frac{\hat{p}_{04}-\hat{p}_{05}}{\hat{p}_{04}+\hat{p}_{05}} \end{bmatrix} \tag{A.13}$$

$$B = \kappa_d \begin{bmatrix} T_{01}\frac{\hat{p}_{0_1}-\hat{p}_{0_2}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_2}-\hat{p}_{0_3}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_1}+\hat{p}_{0_2}} \\ 0 & T_{02}\frac{\hat{p}_{0_2}-\hat{p}_{0_3}}{\hat{p}_{0_2}+\hat{p}_{0_3}} & 2T_{02}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_2}+\hat{p}_{0_3}} & 2T_{02}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_2}+\hat{p}_{0_3}} \\ 0 & 0 & T_{03}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_3}+\hat{p}_{0_4}} & 2T_{03}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_3}+\hat{p}_{0_4}} \\ 0 & 0 & 0 & T_{04}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_4}+\hat{p}_{0_5}} \end{bmatrix} \tag{A.14}$$

$$S = \begin{bmatrix} \hat{p}_{0_1}-\hat{p}_{0_2} & \hat{p}_{0_2}-\hat{p}_{0_3} & \hat{p}_{0_3}-\hat{p}_{0_4} & \hat{p}_{0_4}-\hat{p}_{0_5} \\ 0 & \hat{p}_{0_2}-\hat{p}_{0_3} & \hat{p}_{0_3}-\hat{p}_{0_4} & \hat{p}_{0_4}-\hat{p}_{0_5} \\ 0 & 0 & \hat{p}_{0_3}-\hat{p}_{0_4} & \hat{p}_{0_4}-\hat{p}_{0_5} \\ 0 & 0 & 0 & \hat{p}_{0_4}-\hat{p}_{0_5} \end{bmatrix} \tag{A.15}$$

The semi-implicit scheme is computed in spectral space because it is linear, where n in Eqs. (A.12) and (A.13) is the wave number. The above matrixes are indexed by following way

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \tag{A.16}$$

Appendix B
Matrixes for a semi-implicit scheme for specific hybrid coordinates

From Eqs. (7.11) and (7.12), the matrixes or vectors can be shown as

$$\mathrm{H}_{k1}^{+} p_s = \frac{n(n+1)R_d}{a^2}\left[\sum_{i=1}^{k-1}\frac{4T_{0i}\left(\hat{B}_i\hat{p}_{0i+1}-\hat{B}_{i+1}\hat{p}_{0i}\right)}{\left(\hat{p}_{0i}+\hat{p}_{0i+1}\right)^2}+\frac{T_{0k}\left(\left(\hat{p}_{0k}+3\hat{p}_{0k+1}\right)\hat{B}_k-\left(\hat{p}_{0k}-\hat{p}_{0k+1}\right)\hat{B}_{k+1}\right)}{\left(\hat{p}_{0k}+\hat{p}_{0k+1}\right)^2}\right]p_s \tag{B.1}$$

$$\begin{aligned}\mathrm{A}_{ki}^{+} T_{vi} = \frac{n(n+1)R_d}{a^2}\Bigg\langle & \frac{T_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\left[\underline{\underline{\hat{C}_{0k}T_{vk-1}}}+\left(\underline{\underline{\hat{C}_{0k}}}+\overline{\overline{\hat{C}_{0k+1}}}\right)T_{vk}+\overline{\overline{\hat{C}_{0k+1}T_{vk+1}}}\right] \\ & +\sum_{i=1}^{k-1}\frac{4T_{0i}}{\left(\hat{p}_{0i}+\hat{p}_{0i+1}\right)^2}\left[\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}T_{vi-1}}}+\left(\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}}}-\overline{\overline{\hat{C}_{0i+1}\hat{p}_{0i}}}\right)T_{vi}-\overline{\overline{\hat{C}_{0i+1}\hat{p}_{0i}T_{vi+1}}}\right] \\ & +\frac{2T_{0k}}{\left(\hat{p}_{0k}+\hat{p}_{0k+1}\right)^2}\left[\underline{\underline{\hat{C}_{0k}\hat{p}_{0k+1}T_{vk-1}}}+\left(\underline{\underline{\hat{C}_{0k}\hat{p}_{0k+1}}}-\overline{\overline{\hat{C}_{0k+1}\hat{p}_{0k}}}\right)T_{vk}-\overline{\overline{\hat{C}_{0k+1}\hat{p}_{0k}T_{vk+1}}}\right]+\sum_{i=1}^{k-1}2\frac{\hat{p}_{0i}-\hat{p}_{0i+1}}{\hat{p}_{0i}+\hat{p}_{0i+1}}T_{vi}+\frac{\hat{p}_{0k}-\hat{p}_{0k+1}}{\hat{p}_{0k}+\hat{p}_{0k+1}}T_{vk}\Bigg\rangle\end{aligned} \tag{B.2}$$

$$\mathrm{B}_{ki} D_i^{*} = \frac{\kappa_d T_{0k}}{\hat{p}_{0k}+\hat{p}_{0k+1}}\left(\sum_{i=k}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})+\sum_{i=k+1}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})\right)D_i^{*} \tag{B.3}$$

$$S_{1i} D_i^{*} = \sum_{i=1}^{K}(\hat{p}_{0i}-\hat{p}_{0i+1})D_i^{*} \tag{B.4}$$

for k=1,K with boundary indicated by the double underbar and double overbar, respectively. As in Appendix A, K=4 is used for an example. It is easy to see that

$$\mathrm{H}_{k1}^{+} = \mathrm{H}_{ki}\hat{\mathrm{B}}_{i1} \tag{B.5}$$

where H_{ki} in Appendix A, and B_{i1} is following

$$\hat{\mathrm{B}}_{i1} = \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\\ \hat{B}_3\\ \hat{B}_4\end{bmatrix} \tag{B.6}$$

so the vector H^{+} can be summarized as the following

$$\mathrm{H}^{+} = \frac{n(n+1)}{a^2}R_d\begin{bmatrix}\frac{T_{01}(\hat{p}_{01}+3\hat{p}_{02})}{(\hat{p}_{01}+\hat{p}_{02})^2}\hat{B}_1-\frac{T_{01}(\hat{p}_{01}-\hat{p}_{02})}{(\hat{p}_{01}+\hat{p}_{02})^2}\hat{B}_2 \\ \frac{4T_{01}\hat{p}_{02}}{(\hat{p}_{01}+\hat{p}_{02})^2}\hat{B}_1-\left(\frac{4T_{01}\hat{p}_{01}}{(\hat{p}_{01}+\hat{p}_{02})^2}-\frac{T_{02}(\hat{p}_{02}+3\hat{p}_{03})}{(\hat{p}_{02}+\hat{p}_{03})^2}\right)\hat{B}_2-\frac{T_{02}(\hat{p}_{02}-\hat{p}_{03})}{(\hat{p}_{02}+\hat{p}_{03})^2}\hat{B}_3 \\ \frac{4T_{01}\hat{p}_{02}}{(\hat{p}_{01}+\hat{p}_{02})^2}\hat{B}_1-\left(\frac{4T_{01}\hat{p}_{01}}{(\hat{p}_{01}+\hat{p}_{02})^2}-\frac{4T_{02}\hat{p}_{03}}{(\hat{p}_{02}+\hat{p}_{03})^2}\right)\hat{B}_2-\left(\frac{4T_{02}\hat{p}_{02}}{(\hat{p}_{02}+\hat{p}_{03})^2}-\frac{T_{03}(\hat{p}_{03}+3\hat{p}_{04})}{(\hat{p}_{03}+\hat{p}_{04})^2}\right)\hat{B}_3-\frac{T_{03}(\hat{p}_{03}-\hat{p}_{04})}{(\hat{p}_{03}+\hat{p}_{04})^2}\hat{B}_4 \\ \frac{4T_{01}\hat{p}_{02}}{(\hat{p}_{01}+\hat{p}_{02})^2}\hat{B}_1-\left(\frac{4T_{01}\hat{p}_{01}}{(\hat{p}_{01}+\hat{p}_{02})^2}-\frac{4T_{02}\hat{p}_{03}}{(\hat{p}_{02}+\hat{p}_{03})^2}\right)\hat{B}_2-\left(\frac{4T_{02}\hat{p}_{02}}{(\hat{p}_{02}+\hat{p}_{03})^2}-\frac{4T_{03}\hat{p}_{04}}{(\hat{p}_{03}+\hat{p}_{04})^2}\right)\hat{B}_3-\left(\frac{4T_{03}\hat{p}_{03}}{(\hat{p}_{03}+\hat{p}_{04})^2}-\frac{T_{04}(\hat{p}_{04}+3\hat{p}_{05})}{(\hat{p}_{04}+\hat{p}_{05})^2}\right)\hat{B}_4\end{bmatrix} \tag{B.7}$$

or we can use Eq. (6.4a) to get sum of each layer by replacing variable p by B. Matrix A is not easy to figure out, let's separate it into three matrixes

$$A^{+}_{ki}T_{vi} = \frac{n(n+1)R_d}{a^2}\left(A^{1+}_{ki} + A^{2+}_{ki} + A^{3+}_{ki} + A^{4+}_{ki} + A^{5+}_{ki}\right)T_{vi} \tag{B.8}$$

where

$$A^{1+}_{ki}T_{vi} = \frac{T_{0k}}{\hat{p}_{0k} + \hat{p}_{0k+1}}\left[\underline{\underline{\hat{C}_{0k}T_{vk-1}}} + \left(\underline{\underline{\hat{C}_{0k}}} + \overline{\overline{\hat{C}_{0k+1}}}\right)T_{vk} + \overline{\overline{\hat{C}_{0k+1}T_{vk+1}}}\right] \tag{B.9a}$$

$$A^{2+}_{ki}T_{vi} = \sum_{i=1}^{k-1}\frac{4T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}T_{vi-1}}} + \frac{2T_{0k}}{\left(\hat{p}_{0k} + \hat{p}_{0k+1}\right)^2}\underline{\underline{\hat{C}_{0k}\hat{p}_{0k+1}T_{vk-1}}} \tag{B.9b}$$

$$A^{3+}_{ki}T_{vi} = \sum_{i=1}^{k-1}\frac{4T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\left(\underline{\underline{\hat{C}_{0i}\hat{p}_{0i+1}}} - \hat{C}_{0i+1}\hat{p}_{0i}\right)T_{vi} + \frac{2T_{0k}}{\left(\hat{p}_{0k} + \hat{p}_{0k+1}\right)^2}\left(\underline{\underline{\hat{C}_{0k}\hat{p}_{0k+1}}} - \overline{\overline{\hat{C}_{0k+1}\hat{p}_{0k}}}\right)T_{vk} \tag{B.9c}$$

$$A^{4+}_{ki}T_{vi} = -\sum_{i=1}^{k-1}\frac{4T_{0i}}{\left(\hat{p}_{0i} + \hat{p}_{0i+1}\right)^2}\hat{C}_{0i+1}\hat{p}_{0i}T_{vi+1} - \frac{2T_{0k}}{\left(\hat{p}_{0k} + \hat{p}_{0k+1}\right)^2}\overline{\overline{\hat{C}_{0k+1}\hat{p}_{0k}T_{vk+1}}} \tag{B.9d}$$

$$A^{5+}_{ki}T_{vi} = \sum_{i=1}^{k-1}2\frac{\hat{p}_{0i} - \hat{p}_{0i+1}}{\hat{p}_{0i} + \hat{p}_{0i+1}}T_{vi} + \frac{\hat{p}_{0k} - \hat{p}_{0k+1}}{\hat{p}_{0k} + \hat{p}_{0k+1}}T_{vk} \tag{B.9e}$$

with boundary conditions k=1 and k=K. Then it is easy to figure out, and B9e is the same as in Appendix A. Then the 4x4 example matrixes of Eq. (B.9) can be expanded as

$$A^{1+} = \begin{bmatrix} \frac{T_{01}\hat{C}_{02}}{\hat{p}_{01} + \hat{p}_{02}} & \frac{T_{01}\hat{C}_{02}}{\hat{p}_{01} + \hat{p}_{02}} & 0 & 0 \\ \frac{T_{02}\hat{C}_{02}}{\hat{p}_{02} + \hat{p}_{03}} & \frac{T_{02}\left(\hat{C}_{02} + \hat{C}_{03}\right)}{\hat{p}_{02} + \hat{p}_{03}} & \frac{T_{02}\hat{C}_{03}}{\hat{p}_{02} + \hat{p}_{03}} & 0 \\ 0 & \frac{T_{03}\hat{C}_{03}}{\hat{p}_{03} + \hat{p}_{04}} & \frac{T_{03}\left(\hat{C}_{03} + \hat{C}_{04}\right)}{\hat{p}_{03} + \hat{p}_{04}} & \frac{T_{03}\hat{C}_{04}}{\hat{p}_{03} + \hat{p}_{04}} \\ 0 & 0 & \frac{T_{04}\hat{C}_{04}}{\hat{p}_{04} + \hat{p}_{05}} & \frac{T_{04}\hat{C}_{04}}{\hat{p}_{04} + \hat{p}_{05}} \end{bmatrix} \tag{B.10}$$

$$A^{2+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{2T_{02}\hat{C}_{02}\hat{p}_{03}}{\left(\hat{p}_{02} + \hat{p}_{03}\right)^2} & 0 & 0 & 0 \\ \frac{4T_{02}\hat{C}_{02}\hat{p}_{03}}{\left(\hat{p}_{02} + \hat{p}_{03}\right)^2} & \frac{2T_{03}\hat{C}_{03}\hat{p}_{04}}{\left(\hat{p}_{03} + \hat{p}_{04}\right)^2} & 0 & 0 \\ \frac{4T_{02}\hat{C}_{02}\hat{p}_{03}}{\left(\hat{p}_{02} + \hat{p}_{03}\right)^2} & \frac{4T_{03}\hat{C}_{03}\hat{p}_{04}}{\left(\hat{p}_{03} + \hat{p}_{04}\right)^2} & \frac{2T_{04}\hat{C}_{04}\hat{p}_{05}}{\left(\hat{p}_{04} + \hat{p}_{05}\right)^2} & 0 \end{bmatrix} \tag{B.11}$$

$$
A^{3+}=\begin{bmatrix}
\frac{2T_{01}\left(\hat{C}_{01}\hat{p}_{02}-\hat{C}_{02}\hat{p}_{01}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 & 0 \\
\frac{4T_{01}\left(\hat{C}_{01}\hat{p}_{02}-\hat{C}_{02}\hat{p}_{01}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{2T_{02}\left(\hat{C}_{02}\hat{p}_{03}-\hat{C}_{03}\hat{p}_{02}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 & 0 \\
\frac{4T_{01}\left(\hat{C}_{01}\hat{p}_{02}-\hat{C}_{02}\hat{p}_{01}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{4T_{02}\left(\hat{C}_{02}\hat{p}_{03}-\hat{C}_{03}\hat{p}_{02}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{2T_{03}\left(\hat{C}_{03}\hat{p}_{04}-\hat{C}_{04}\hat{p}_{03}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & 0 \\
\frac{4T_{01}\left(\hat{C}_{01}\hat{p}_{02}-\hat{C}_{02}\hat{p}_{01}\right)}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{4T_{02}\left(\hat{C}_{02}\hat{p}_{03}-\hat{C}_{03}\hat{p}_{02}\right)}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{4T_{03}\left(\hat{C}_{03}\hat{p}_{04}-\hat{C}_{04}\hat{p}_{03}\right)}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} & \frac{2T_{04}\left(\hat{C}_{04}\hat{p}_{05}-\hat{C}_{05}\hat{p}_{04}\right)}{\left(\hat{p}_{04}+\hat{p}_{05}\right)^2}
\end{bmatrix} \tag{B.12}
$$

$$
A^{4+}=\begin{bmatrix}
0 & \frac{-2T_{01}\hat{C}_{02}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & 0 & 0 \\
0 & \frac{-4T_{01}\hat{C}_{02}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-2T_{02}\hat{C}_{03}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & 0 \\
0 & \frac{-4T_{01}\hat{C}_{02}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-4T_{02}\hat{C}_{03}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{-2T_{03}\hat{C}_{04}\hat{p}_{03}}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2} \\
0. & \frac{-4T_{01}\hat{C}_{02}\hat{p}_{01}}{\left(\hat{p}_{01}+\hat{p}_{02}\right)^2} & \frac{-4T_{02}\hat{C}_{03}\hat{p}_{02}}{\left(\hat{p}_{02}+\hat{p}_{03}\right)^2} & \frac{-4T_{03}\hat{C}_{04}\hat{p}_{03}}{\left(\hat{p}_{03}+\hat{p}_{04}\right)^2}
\end{bmatrix} \tag{B.13}
$$

$$
A^{5+}=\begin{bmatrix}
\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 0 & 0 & 0 \\
2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & \frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & 0 & 0 \\
2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 2\frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & \frac{\hat{p}_{03}-\hat{p}_{04}}{\hat{p}_{03}+\hat{p}_{04}} & 0 \\
2\frac{\hat{p}_{01}-\hat{p}_{02}}{\hat{p}_{01}+\hat{p}_{02}} & 2\frac{\hat{p}_{02}-\hat{p}_{03}}{\hat{p}_{02}+\hat{p}_{03}} & 2\frac{\hat{p}_{03}-\hat{p}_{04}}{\hat{p}_{03}+\hat{p}_{04}} & \frac{\hat{p}_{04}-\hat{p}_{05}}{\hat{p}_{04}+\hat{p}_{05}}
\end{bmatrix} \tag{B.14}
$$

The matrix B is not changed, and can be copied from Appendix A as the following.

$$
B=\kappa_d\begin{bmatrix}
T_{01}\frac{\hat{p}_{0_1}-\hat{p}_{0_2}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_2}-\hat{p}_{0_3}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_1}+\hat{p}_{0_2}} & 2T_{01}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_1}+\hat{p}_{0_2}} \\
0 & T_{02}\frac{\hat{p}_{0_2}-\hat{p}_{0_3}}{\hat{p}_{0_2}+\hat{p}_{0_3}} & 2T_{02}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_2}+\hat{p}_{0_3}} & 2T_{02}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_2}+\hat{p}_{0_3}} \\
0 & 0 & T_{03}\frac{\hat{p}_{0_3}-\hat{p}_{0_4}}{\hat{p}_{0_3}+\hat{p}_{0_4}} & 2T_{03}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_3}+\hat{p}_{0_4}} \\
0 & 0 & 0 & T_{04}\frac{\hat{p}_{0_4}-\hat{p}_{0_5}}{\hat{p}_{0_4}+\hat{p}_{0_5}}
\end{bmatrix} \tag{B.15}
$$

Then the vector S for surface pressure is the following

$$S = \left[\hat{p}_{0_1} - \hat{p}_{0_2} \quad \hat{p}_{0_2} - \hat{p}_{0_3} \quad \hat{p}_{0_3} - \hat{p}_{0_4} \quad \hat{p}_{0_4} - \hat{p}_{0_5}\right] \tag{B.16}$$

Note that, this matrix has the same sequence as the one in Appendix A.

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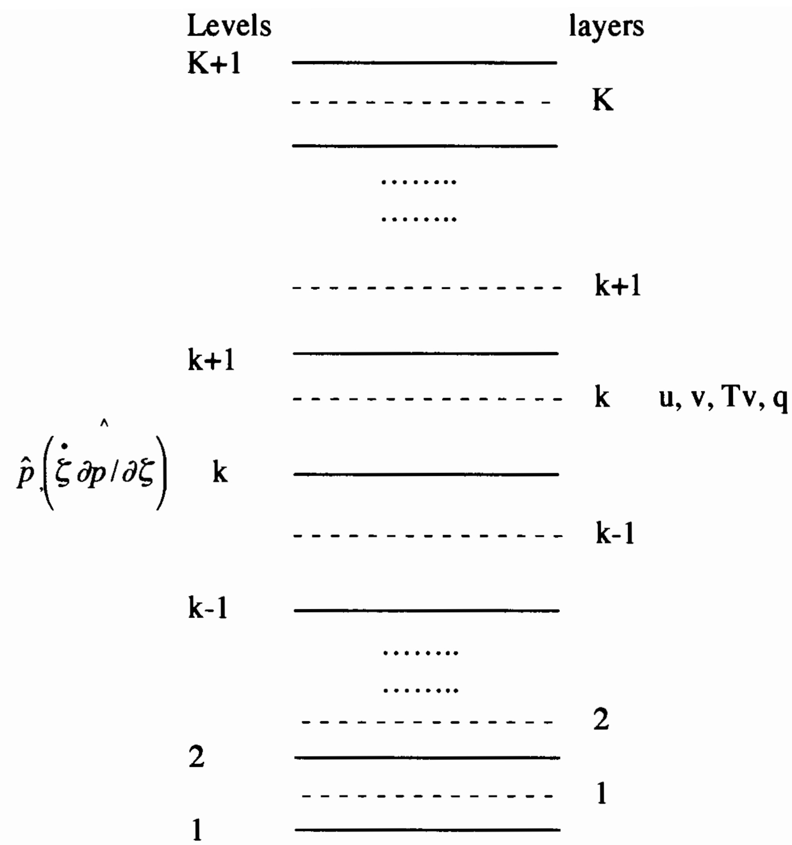

Fig. 1 The vertical grid structure with layers and levels. Integers are used to index layers and levels; variables marked with hats are on levels and without hats are on layers.